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WorldMathBook

English

For high school and beyond

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Content:

Part 3. Differentiation and Integration

Preface, WorldMath - English

Mathematics is our most accurate science.

Mathematics is a beautiful science.

Some study mathematics alone, but most people use it as a tool for physics, biology, medicine, engineering science, economy, ……, for everything.

For high school and more. We start with the four basic arithmetic operations, and finish in the first or second semester of the study for bachelor or candidate.

The language is clear, understanding is in focus, technical terms are explained.

There is also an exercise book with problems and proposed solutions.

The book is independent of which formula collection is used.

The book is also independent of using a calculator or a calculation program.

And one more thing. Mathematics is not becoming more and more complicated as we go along. That is my personal experience, and I see it with the students too. The next step is not harder.

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Tom Pedersen, January 2024.

Part 1. Basics

Number system

We use the decimal system (10s system). Probably because we have 10 fingers. We state, that our number system has the base number 10.

In ancient times the Greeks also used a 10s system, and they were able to calculate, but their number characters were different and unfortunately they had no character for zero. This made their system difficult and it was only mastered by few.

The Romans also used a 10s system and still they had no character for zero. Their characters consisted of letters (for example was 12 written this way: XII). Roman numerals are still used for indicating the year of a statue or the like. They could not find practical methods of addition and subtraction, and it became even more complicated within multiplication and division.

In the Middle Ages the 10s system was combined with Arab characters (originally from India), and a sign, 0, was added to describe nothing. Today the characters are: 0 1 2 3 4 5 6 7 8 9. And after using these ten characters we can write ten new numbers by putting 1 in front: 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, - then 2 in front: 20, 21, 22, 23, 24, 25, 26, 27, 28, 29 - , etc. When we reach 99 we start again using the same characters with 1 in front and two characters after: 100, 101, 102, etc. So, we only use ten characters and their position decide if we have ones, tens, hundreds, thousands, etc. Now we are getting somewhere, and today we have fine tools for addition, subtraction, multiplication, and division (The four basic arithmetic operations).

If you will practice the logic in our number system, I suggest to watch a measuring tape. It is also suitable for practicing the four basic arithmetic operations.

Numbers can be positive, for instance +5 where we can omit the sign + and just write 5. It cannot be misunderstood.

Numbers can also be negative, for instance -5 where we cannot omit the sign: -

If we only want the magnitude of a number we place the number between two straight parenthesis:

$|5| = 5 \qquad |-5| = 5 \qquad |-8| = 8$

We call it *the numerical value* of that number.

numerical value = magnitude of the number

Also, let us briefly describe the number system from ancient Mesopotamia. It is still in use although we don´t think about it. They had two significant numbers: 6 and 60. It is not clear why they chose the significant number 6, but they probably thought that it was too small so they multiplied by 10 (probably due to the ten fingers) and had the base number: 60.

At equinox they stated 6 hours from sunrise till noon and 6 hours from noon till sunset. The night is equally long giving us 24 hours.

An hour is coarse so we divide it with their base number, 60, and get one minute. If we want it finer, we divide by 60 a second time and get one second. In modern times we have divided even more, but this time we use the 10s system! Then we have a tenth, a hundredth, etc. of a second.

In mathematics angles are measured in angle degrees stating one round to be 360°. 360 is found by multiplying the two significant numbers: $6 \cdot 60 = 360$.

Arithmetic is Ancient Greek and means the doctrine of numbers.

The four basic arithmetic operations

The four basic arithmetic operations and their calculation symbols are

1. to add, plus $\quad +$

2. to subtract, minus $\quad -$

3. to multiply, dot $\quad \cdot$

4. to divide, two dots $\quad :$ or a fraction line —

1. Sum

We add by placing the numbers above one another. Ones above ones, tens above tens, hundreds above hundreds, and so on.

$$\begin{array}{r} {}^{1} \\ 117 \\ +\ 14 \\ \hline \underline{\underline{131}} \end{array}$$

First the ones: 7 + 4 gives 11, then at the result bottom line we write the ones, here 1, and the tens are written above the other tens.

then the tens: 1 of before + 1 + 1 gives 3 and is written in the result.

and then the hundreds: 1 + 0 (there is nothing in front of the figures 1 and 4) gives 1 which is written in the result.

answer: 131

2. Difference

We withdraw by placing the numbers above one another. Ones above ones, tens above tens, hundreds above hundreds, and so on.

$$\begin{array}{r} {\scriptstyle 10\ 10\,} \\ \cancel{1}\cancel{1}4 \\ -\ 1\,7 \\ \hline 9\,7 \end{array}$$

First the ones: 4 - 7 which we cannot, so we borrow ten and write it uppermost. These 10 + 4 gives 14. Now we can say 14 - 7 which gives 7, and write it in the result.

then the tens: Uppermost of the tens was 1, but we borrowed it, so now it is 0. 0 - 1 cannot be done so we borrow ten from the hundreds and write it uppermost. It becomes 10 because 100 is ten times bigger than 10. The borrowed 10 - 1 gives 9, and we write it in the result.

And finally the hundreds which used to have the figure 1, but since we borrowed it, we now have 0. 17 does not have any hundreds, so it renders 0 - 0 = 0 which is not written.

answer: 97

2a.

What happens if we want to withdraw a big number from a small number?

It can be done, though we do not have a technique for it. So we flip the numbers and find the big number minus the small number. Then we flip the numbers back again and put a minus in front of the result:

$$\left.\begin{array}{r} 17 \\ -114 \end{array}\right\} \curvearrowright \left.\begin{array}{r} 114 \\ -17 \\ \hline 97 \end{array}\right\} \curvearrowright \underline{\underline{-97}}$$

answer: -97

Negative numbers also exist. No problem. For example may -97 mean there is a deficit of 97.

The curved arrow is a way to show that something is changed, it means: transferred to.

3. Product

We multiply two numbers by placing them next to one another with a dot symbol in between.

First we multiply the *ones of the first number* with the ones of the second number, then *ones* with tens, *ones* with hundreds, etc.

Then we multiply *the tens of the first number* with the ones of the second number, then *tens* with tens, *tens* with hundreds, etc.

And so on….

$$\begin{array}{r} ^{1} \\ 32 \cdot 741 \\ \hline 1482 \\ 22230 \\ \hline \underline{\underline{23712}} \end{array}$$

2 · 1 gives 2 which is written at the ”ones place” below the line.
2 · 4 gives 8 which is written at the ”tens place” below the line.
2 · 7 gives 14 - where 4 is written at the “hundreds place” and 1 at the “thousands place” below the line.

Now *the tens in the first number*: Since we are multiplying by a tens, we start by writing 0 at the ones place below 1482. Then we multiply: $3 \cdot 1$ gives 3 which is written at the tens place. $3 \cdot 4$ gives 12, we write 2 and save 1 as a small figure above the figure of 7. $3 \cdot 7$ gives 21, and we remember to add 1 giving 22, which we just write because there is no more to multiply with. Finally, we add 1482 and 22230 to render 23712.

3a.

If one of the numbers is a decimal point, we multiply as if nothing has happened, and put a comma/dot in the result in the same position as the number we started with:

$$
\begin{array}{r}
32 \cdot 7.41 \\
\hline
1482 \\
22230 \\
\hline
\underline{\underline{237.12}}
\end{array}
$$

If both numbers are decimal points, we multiply as if nothing has happened and put a comma/point in the result in the position for the first number + the position for the second number we began with:

$$
\begin{array}{r}
3.2 \cdot 7.41 \\
\hline
1482 \\
22230 \\
\hline
23.712
\end{array}
$$

Here we have $1 + 2 = 3$ figures after the comma/point.

4. Division

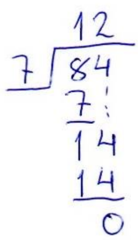
If we are to divide 84 by 7, we write:

84:7 in order to keep the height in a body text.

Or better:

$\frac{84}{7}$ as we prefer it in mathematics.

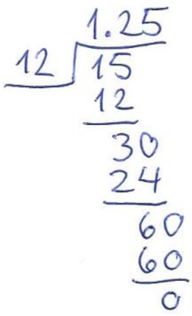
For calculation we arrange it like this:



and we have: 8 divided by 7 gives 1, which is written above. 8-7 gives 1, which is written below and renders a surplus of 1 tens. Now we drag the figure 4 to stand next to the figure 1 so it becomes 14. 14 divided by 7 gives 2, which is written above. 7·2 is 14. 14 minus 14 is zero, so it adds up, and the answer is 12.

4a.

Some times the answer is not a whole number:

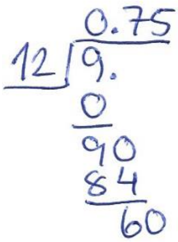


15 divided by 12 gives 1, which is written above. 1·12 gives 12. 15-12 gives 3. So - we have a surplus of 3 and no more figures.

Now we expand 15 to become 15,0000 (with as many zeros as needed). Thus we can put a comma/point to the answer and drag 0 to stand next to the figure 3. Then 30 divided by 12 gives 2, which is written above in the result. 2·12 gives 24. 30-24 is 6. The next 0 is dragged and we have 60. 60 divided by 12 gives 5, which is written in the result. 5·12 gives 60. 60-60 is 0, and we are done. The answer is 1,25.

4b.

And when dividing a small number by a big number:

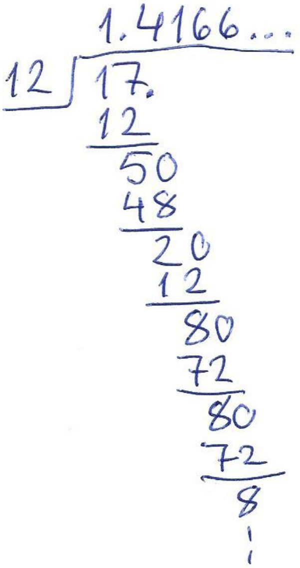


9 divided by 12 can be done 0 times, which is written in the result. Then we expand 9 to become 9.000. We put a comma/point in the result and continue.

The answer is 0.75.

4.c

And when it does not add up



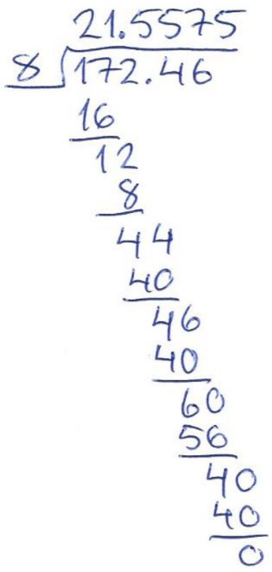
the answer can never be precise. We have to choose how many figures after the comma/point are needed. A decimal is the technical term for a figure after the comma/point. Decimal means "a tenth number". The answer is written 1.4166… the dots show that it continues.

If we want a precise number, we shall not carry out the division calculation at all. We just have to leave the fraction unchanged:

$\frac{17}{12}$ that is precise.

4d.

If we divide a decimal number by a whole number, we put a comma/point in the result when we drag the first decimal



Here the calculation adds up since we end with 0. So, the answer is a precise decimal number: 21.5575

Theory

Finally a few practical remarks:

We can always divide by 1. For instance we can write 3 as a fraction: $\frac{3}{1}$ which surely equals 3.

We can also always multiply by 1. For instance we can write 3 as $3 \cdot 1$ which surely equals 3.

Fractions (quotients)

$\frac{1}{2}$ means 1 divided by 2 *or* 1 out of 2 *or* 1 in proportion with 2. 1 is in the *numerator* and 2 is in the *denominator*. The line in between is called the fraction line.

If we are to share a cake equally we get $\frac{1}{2}$ each *or* we get 1 out of 2 pieces each *or* we get 1 piece in proportion with 2 pieces.

We may multiply the numerator and the denominator with the same number *or* letter *or* other - except 0. We may not multiply by something different since we still need to keep the proportion between numerator and denominator. In this example the denominator must stay twice as big as the numerator.

If we multiply by 3, we get: $\frac{1}{2} = \frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6}$

If we multiply by -3, we get: $\frac{1}{2} = \frac{1 \cdot (-3)}{2 \cdot (-3)} = \frac{-3}{-6}$

If we multiply by a we get: $\frac{1}{2} = \frac{1 \cdot a}{2 \cdot a} = \frac{1a}{2a}$

If we multiply by (0,1·2 - 7) we get

$$\frac{1}{2} = \frac{1 \cdot (0.1 \cdot 2 - 7)}{2 \cdot (0.1 \cdot 2 - 7)} = \frac{1(0.1 \cdot 2 - 7)}{2(0.1 \cdot 2 - 7)}$$

In these four examples, we have extended the fraction.

We may also divide the numerator and the denominator with the same number *or* letter *or* other - except 0.

If we divide $\frac{3}{6}$ by 3 in numerator and denominator we get $\frac{1}{2}$, and we are back.

If we divide $\frac{1(0.1\cdot 2-7)}{2(0.1\cdot 2-7)}$ by (0,1·2 - 7) in numerator and denominator we get $\frac{1}{2}$, and we are back.

We have shortened the fraction.

When multiplying a number (or other) by a fraction, we multiply number and numerator. For instance

$5\cdot\frac{1}{2}=\frac{5}{2}$ or $a\cdot\frac{3}{6}=\frac{3a}{6}$

or $(0.1\cdot 2-7)\cdot\frac{3}{6}=\frac{3\,(0.1\cdot 2-7)}{6}$

When multiplying a fraction by a fraction, we multiply numerator by numerator and denominator by denominator. For instance

$\frac{3}{6}\cdot\frac{5}{2}=\frac{15}{12}$ or $\frac{3}{6}\cdot\frac{-5}{2}=\frac{-15}{12}$

or $\frac{3}{6}\cdot\frac{5(0.1\cdot 2-7)}{2}=\frac{3\cdot 5\cdot(0.1\cdot 2-7)}{6\cdot 2}=\frac{15(0.1\cdot 2-7)}{12}$

It is a little harder to divide a fraction by a fraction. For instance $\frac{1}{2}$ divided by $\frac{1}{4}$

$\frac{\frac{1}{2}}{\frac{1}{4}}$ here it is important to write it in a way that clearly shows what is to be divided by what. It must not be misunderstood, so we write

$\dfrac{\frac{1}{2}}{\frac{1}{4}}$ now we can clearly see, that $\frac{1}{2}$ is to be divided by $\frac{1}{4}$

And now it becomes a bit more difficult. We can see that $\frac{1}{2}$ is twice as big as $\frac{1}{4}$ so the answer has to be 2. Therefore, the rule says that we divide a fraction by a fraction - by instead multiplying one fraction by the inverse of the other fraction. The inverse of $\frac{1}{4}$ is $\frac{4}{1}$ so

$$\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{1}{2} \cdot \frac{4}{1} = \frac{4}{2} = 2$$

We can also see it this way:

We multiply both numerator and denominator by 4 and get

$$\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{\frac{4}{2}}{\frac{4}{4}} = \frac{2}{1} = 2$$

And the same rule applies if the numerator is not a fraction

$$\frac{(0.1\cdot 2 - 7)}{\frac{1}{4}} = 4(0.1\cdot 2 - 7)$$

or $\frac{318.27}{\frac{1}{4}} = 4 \cdot 318.27$

Examples

1.

The purpose of shortening a fraction usually is simplification

$\frac{\frac{1}{2} \cdot 6}{\frac{1}{4} \cdot 8}$ may be shortened as $\frac{3}{2}$

$\dfrac{(-\frac{1}{2}) \cdot 6}{\frac{1}{4} \cdot 8}$ may be shortened as $\dfrac{-3}{2} = -\dfrac{3}{2}$

It does not matter if minus is before the whole numerator or before the whole fraction.

2.

The purpose of extending a fraction usually is the wish for a certain denominator for further calculation, in particular if we are adding fractions

$\frac{1}{2} + \frac{1}{3}$ here we need to find a common denominator. One can always find a common denominator by multiplying the two denominators, here: $2 \cdot 3 = 6$, so we change both fractions into sixth - which we shorten afterwards

$$\frac{1}{2} + \frac{1}{3} = \frac{1 \cdot 3}{2 \cdot 3} + \frac{1 \cdot 2}{3 \cdot 2} = \frac{5}{6}$$

3. And a mix

$$\frac{(-\frac{1}{2}) \cdot 6}{\frac{1}{4} \cdot 8} + \frac{1}{3} = \frac{3\left(-\frac{1}{2}\right) \cdot 6}{3(\frac{1}{4} \cdot 8)} + \frac{1(\frac{1}{4} \cdot 8)}{3(\frac{1}{4} \cdot 8)}$$

where the common denominator is found by multiplying the two denominators. Then we can make a common fraction

$$\frac{3\left(-\frac{1}{2}\right) \cdot 6 + 1(\frac{1}{4} \cdot 8)}{3(\frac{1}{4} \cdot 8)} = \frac{(-9) + (2)}{(6)} = \frac{-7}{6}$$

In this calculation, we multiplied an entity within a parenthesis by a number. We will look more into this in a following chapter.

Percent

Percent means ”out of a hundred”, which means a fraction with 100 as the denominator.

$\frac{1}{2}$ means 1 out of 2. If we multiply by 50 in the numerator and denominator we get $\frac{50}{100}$ or 50 out of 100 or 50%. In brief:

$$\frac{50}{100} = 50\%$$

Examples

$$\frac{1}{5} = \frac{20 \cdot 1}{20 \cdot 5} = \frac{20}{100} = 20\%$$

$$\frac{1}{8} = \frac{12{,}5 \cdot 1}{12{,}5 \cdot 8} = \frac{12{,}5}{100} = 12{,}5\%$$

$$\frac{1}{4} = \frac{25}{100} = 25\%$$

and as a decimal number

$$\frac{1}{2} = \frac{50 \cdot 1}{50 \cdot 2} = \frac{50}{100} = 50\% = 0{,}5$$

$$\frac{1}{4} = \frac{25 \cdot 1}{25 \cdot 4} = \frac{25}{100} = 25\% = 0{,}25$$

$$\frac{3}{4} = \frac{25 \cdot 3}{25 \cdot 4} = \frac{75}{100} = 75\% = 0{,}75$$

$$\frac{3}{8} = \frac{12{,}5 \cdot 3}{12{,}5 \cdot 8} = \frac{37{,}5}{100} = 37{,}5\% = 0{,}375$$

Percent is out of a hundred. A decimal number is out of one.

1 is one whole. 100% is also one whole.

$$1 = \frac{100}{100} = 100\%$$

2.

Yesterday a certain dress cost 200 pounds. Today it has risen to 225 pounds. What is the rise in %?

200 pounds corresponds with 100%.

The rise is 225 - 200 = 25 pounds, which must be seen in proportion with the 200 pounds:

$\frac{25}{200} = 0.125 = 12.5\%$ which is the answer

3.

Yesterday a certain dress cost 200 pounds. Today the price has dropped to 175 pounds. What is the price reduction in %?

200 pounds corresponds with 100%.

The reduction is 200 - 175 = 25 pounds, which must be seen in proportion with the 200 pounds:

$\frac{25}{200} = 0.125 = 12.5\%$ which is the answer

The information could be given as: Today -12.5% for this dress.

4.

The price for a certain machine is 1000 pounds without VAT.

1000 pounds corresponds to 100%. Inclusive of 25% VAT the price is:

$1.25 \cdot 1000 = 1250$ pounds which is the answer

or

$100\% + 25\% = 1000 + 0.25 \cdot 1000 = 1250$ pounds

5.

Another machine costs 1000 pounds with VAT.

1000 pounds now corresponds to 125%. Exclusive of 25% VAT the price is:

$\frac{1000}{1.25}$ = 800 pounds which is the answer

We can confirm by saying

100% + 25% = 800 + 0.25·800 = 1000 pounds

6.

What is the percentage of 347 out of 376?

$\frac{347}{376}$ = ca. 0.9229 = ca. 92.3%

Percentage point

If we have woven 20% of a whole blanket in March and 25% of the whole blanket in April we have increased by 5 percentage point (5% point).

Or: change = end - start = 25% - 20% = 5% point

Or: $\frac{25}{100} - \frac{20}{100} = \frac{5}{100}$ = 5% point

Thus percentage point expresses the change/the difference/the increase or decrease.

+5% point is an increase/growth.

-5% point is a decrease/drop.

Calculation with letters (algebra)

If we do not know the number of something, we call it an unknown quantity and write a letter instead. That is *algebra*.

All rules of calculation are the same. That goes for the four basic arithmetic operations as well as for other types of calculation, which we will meet later.

The technical term, algebra, is from Latin.

Many technical terms are from Latin or ancient Greek and serves as a common language in most science regardless of which language we use otherwise. Also, many technical terms are in English which is understood and spoken by many.

For unknown quantities we use small letters (a, b, c, etc.) and capital letters (D, E, F, etc.) of our alphabet. But often, that is not enough, so we also use small (α, β, γ, etc.) and capital (Δ, Θ, Σ, etc.) letters from ancient Greek. It may be the name of a line, an angle or other.

We may also use letters as abbreviations. We will see that later.

Examples

1.

$a + a = 2a$	$\alpha + 2\alpha = 3\alpha$	$- \alpha + 2\alpha = \alpha$
$2 \cdot a = 2a$	$a \cdot b = ab$	$a \cdot b \cdot c = abc$

We may omit the multiplication dot if it cannot be misunderstood. We like to do things briefly.

2.

A + B

cannot be reduced since A may be the number of apples and B the number of pears. That cannot be changed so the answer still is

A + B

and

B - c = B - c $\qquad \frac{a}{b} = \frac{a}{b} \qquad - \frac{3a}{2b} = - \frac{3a}{2b}$

and so on.

3.

$$- \frac{3a}{4b} + \frac{3a-c+2\cdot a}{2b} = - \frac{3a}{4b} + \frac{5a-c}{2b} = - \frac{3a}{4b} + \frac{2\cdot(5a-c)}{2\cdot 2b} = \frac{-3a+10a-2c}{4b}$$

$= \frac{7a-2c}{4b}$ which cannot be simplified.

4.

$$\frac{\frac{1}{2}\cdot 6\cdot a}{\frac{1}{4}\cdot 8a} = \frac{3\cdot a}{2\cdot a} = \frac{3}{2}$$

5.

$$\frac{(-\frac{1}{2})\cdot 6\cdot a}{\frac{1}{4}\cdot 8b} = \frac{-3a}{2b} = - \frac{3a}{2b}$$

6.

$\frac{4x-8y}{2}$

may be split into two fractions where both 4x and -8y must be divided by 2:

$\frac{4x-8y}{2} = \frac{4x}{2} - \frac{8y}{2} =$ 2x - 4y

7.

3x - 3y + $\frac{4x-8y}{4}$

The fraction bothers since we would like to have x alone and y alone. We split the fraction in two

3x - 3y + $\frac{4x}{4} - \frac{8y}{4}$

the minus before 8y is moved to the front of the fraction. We can do that since there is nothing else in the numerator than just 8y.

3x - 3y + x - 2y = 4x - 5y

x alone and y alone. Further reduction is not possible.

Parenthesis

We put a parenthesis around something we want to see as one, - one number, - one entity.

Examples

1.

$4(x + y)$ here $(x + y)$ as one is to be multiplied by 4.

2.

$(x + y) + 4$ here $(x + y)$ as one is added to 4.

3.

$4 - (x + y)$ here $(x + y)$ as one is subtracted from 4.

4.

$$\frac{4}{(x+y)}$$

here 4 is divided by $(x + y)$. $(x + y)$ cannot be separated.

5.

$$\frac{(x+y)}{4}$$

Here $(x + y)$ is to be divided by 4 and may be split, so that x is divided by 4 separately - and y is divided by 4 separately. Thus, two fractions

$$\frac{x}{4} + \frac{y}{4}$$

If we want to lift a parenthesis, we have to make sure that the meaning is unchanged and that you can calculate backwards to obtain the expression you started with.

6.

$4(x + y) = 4x + 4y$ 4 is multiplied into the parenthesis by multiplying x and y separately.

If we calculate backwards from right to left, we put 4 outside the parenthesis.

7.

$(x + y) + 4$ here we have an invisible + before the parenthesis. Mathematicians almost always write things in short, so $(x + y)$ is understood as $+(x + y)$. A plus parenthesis may be lifted with no further change

$(x + y) + 4 = x + y + 4$

We may also calculate from right to left by putting any parenthesis we want to.

8.

If the entity within the parenthesis is negative, we write $-(x + y)$. So, when we lift the parenthesis both x and y are negative

$4 - (x + y) = 4 - x - y$

If we calculate from right to left we put -1 outside the parenthesis. Again, we want to be brief and just write - before the parenthesis. It cannot be misunderstood.

$-1\cdot(x + y) = -1(x + y) = -(x + y)$

9.

$$\frac{4}{(x+y)}$$

here 4 is divided by (x + y). (x + y) cannot be separated. The parenthesis may be removed, it changes nothing, 4 is still to be divided by the sum x+y

$$\frac{4}{(x+y)} = \frac{4}{x+y}$$

10.

$$\frac{(x+y)}{4}$$

Here (x + y) is to be divided by 4 and may be split, so that x is divided by 4 separately - and y is divided by 4 separately. Thus, two fractions

$\frac{x}{4} + \frac{y}{4}$ at the same time, the parenthesis has been lifted

and we may calculate backwards by making a common fraction.

CAS does not have the ability, like that of man, to distinguish between necessary or unnecessary parenthesis. Therefore, we may need to put more parentheses when we use CAS.

Square rules (Remarkable identities)

First a few examples of how to multiply entities in parentheses:

$(2 + a)\cdot(3 + 2a) = 6 + 4a + 3a + 2aa = 6 + 7a + 2aa$

we multiply 2 by 3, then 2 by 2a, a by 3, and finally a by 2a. Eventually we reduce.

$(2 + a)(3 + 2a + b) = 6 + 4a + 2b + 3a + 2aa + ab$

The method is the same. First we multiply 2 by 3, by 2a, by b, then we multiply a by 3, by 2a, by b.

And

$(2 - a)(3 - 2a - b) = 6 - 4a - 2b - 3a + 2aa + ab$ remember the sign

Plus times plus gives plus

Plus times minus gives minus

Minus times plus gives minus

Minus times minus gives plus.

2aa can also be written this way: $2aa = 2a^2$ we say "two times - a to the power of two"

or $b\cdot b = bb = b^2$ b to the power of two (or: b squared)

and now the square rules (remarkable identities):

1. $(a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + b^2 + 2ab$

2. $(a - b)(a - b) = a^2 - ab - ba + b^2 = a^2 + b^2 - 2ab$

3. $(a + b)(a - b) = a^2 - ab + ba - b^2 = a^2 - b^2$

In the first theorem $(a + b)$ is squared and we write $(a + b)^2$. In the second theorem $(a - b)$ is squared and we write $(a - b)^2$. In the third theorem the signs vary, so this is no square.

The brief version is

1. $(a + b)^2 = a^2 + b^2 + 2ab$

2. $(a - b)^2 = a^2 + b^2 - 2ab$

3. $(a + b)(a - b) = a^2 - b^2$

These theorems are used a lot.

Examples

$$\frac{4a^2-9}{4a^2+9-12a} = \frac{(2a+3)(2a-3)}{(2a-3)^2} = \frac{2a+3}{2a-3}$$

$$\frac{3b^2+12-12b}{5b^2-10b} = \frac{3(b^2+4-4b)}{5b^2-10b} = \frac{3(b-2)^2}{5b(b-2)} = \frac{3(b-2)}{5b}$$

and a long reduction:

$$\frac{3}{ab-b^2} + \frac{3}{a^2+ab} - \frac{6}{a^2-b^2} =$$

$$\frac{3}{b(a-b)} + \frac{3}{a(a+b)} - \frac{6}{(a+b)(a-b)} =$$ b put out, a put out, theorem 3

$$\frac{3(a+b)}{b(a+b)(a-b)} + \frac{3(a-b)}{a(a+b)(a-b)} - \frac{6}{(a+b)(a-b)} =$$ prolonged, prolonged, nothing

$$\frac{\frac{3}{b}(a+b)}{(a+b)(a-b)} + \frac{\frac{3}{a}(a-b)}{(a+b)(a-b)} - \frac{6}{(a+b)(a-b)} =$$ b moved, a moved, nothing

$$\frac{\frac{3a}{b}+3+3-\frac{3b}{a}-6}{(a+b)(a-b)} = \frac{\frac{3a}{b}-\frac{3b}{a}}{(a+b)(a-b)} = \frac{\frac{3a^2}{ab}-\frac{3b^2}{ab}}{(a+b)(a-b)} = \frac{3a^2-3b^2}{ab(a^2-b^2)} = \frac{3}{ab}$$

Square root

If we have a number, for instance 4, we can find the positive number that gives 4 when squared. That is 2. We say that the square root of 4 is 2, and we write $\sqrt{4} = 2$

And further: If we have a number, for instance 8, we can find the positive number that gives 8 when raised to the power of 3. That is 2. We say that the third root of 8 is 2, and we write

$$\sqrt[3]{8} = 2$$

and the fourth root of 16 is 2

$$\sqrt[4]{16} = 2$$

The square root and so forth may also be written in another way, as we shall see in the next chapter: "Exponentiation".

Examples

1.

$$\sqrt{16 \cdot 4} = \sqrt{64} = 8$$

or

$$\sqrt{16} \cdot \sqrt{4} = 4 \cdot 2 = 8$$

so

$$\sqrt{16 \cdot 4} = \sqrt{16} \cdot \sqrt{4}$$

2.

$$\frac{\sqrt{16}}{\sqrt{4}} = \frac{4}{2} = 2$$

or

$$\sqrt{\frac{16}{4}} = \sqrt{4} = 2$$

so

$$\frac{\sqrt{16}}{\sqrt{4}} = \sqrt{\frac{16}{4}}$$

Or in letters

3.

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

4.

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

5.

$$\frac{\sqrt{16a}}{\sqrt{25b}} = \frac{4\sqrt{a}}{5\sqrt{b}}$$

Exponentiation

There is a shorter way of writing a times a

$a \cdot a = aa = a^2$

a is the base number, 2 is called the exponent and a^2 together is: a raised to the power of 2.

More examples

$aaaaaaaaaaaaa = a^{13}$ $bbbb = b^4$

or in numbers:

$7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^6$ $9 \cdot 9 \cdot 9 = 9^3$

With letters we may omit the multiplication sign, it cannot be misunderstood. However, with figures we cannot omit the sign, because 999 means nine hundred and ninety nine.

Especially important and widely used are the powers of 10:

$10 \cdot 10 \cdot 10 = 10^3$ $10 \cdot 10 \cdot 10 \cdot 10 = 10^4$ etc.

There are three advantages of using exponentiation: They are well suited for very big and very small numbers, they are easier to calculate with (once one has got used to it), and they are almost indispensable in differential- and integral calculus as we will see later.

If we have a fraction like $\frac{10 \cdot 10 \cdot 10}{1}$, -we can write $\frac{10^3}{1} = 10^3$

Understood a + before 3. Thus $10^{+3} = 10^3$

If we have a fraction like $\frac{1}{10 \cdot 10 \cdot 10}$, -we can write $\frac{1}{10^3} = 10^{-3}$

Here the sign minus shows that 10^3 is located in the denominator.

+ before the exponent means location: numerator, and - before the exponent means location: denominator.

Now, instead of one thousand we can write

$1000 = 10^3$

and instead of one million we can write

$1\,000\,000 = 10^6$

Avogadro's number (physics, chemistry) is approximately: $6 \cdot 10^{23}$

Planck's constant (physics) is approximately: $6{,}63 \cdot 10^{-34}$

These numbers would be very tedious to write without the use of exponentiation.

Examples

1.

$10^2 \cdot \mathit{10^3} = 10 \cdot 10 \cdot \mathit{10} \cdot \mathit{10} \cdot \mathit{10} = 10^5$

We could have calculated this by just adding the exponents

$10^2 \cdot 10^3 = 10^5$

2.

$$\frac{1}{10^2 \cdot 10^3} = \frac{1}{10^5} = \frac{10^{-5}}{1} = 10^{-5}$$

We could have calculated this by adding the exponents in the denominator - and moving the power number to the numerator with a minus before the exponent.

3.

$\frac{10^4 \cdot 10^{-2} \cdot 10^5}{10^2 \cdot 10^3} = 10^2 = 100$

Exponents in the numerator: 4-2+5 = 7

Exponents in the denominator: 2+3 = 5 moved up -5

Calculation of the exponents: 7-5 = 2 that means $10^2 = 100$.

4.

$10^{½} \cdot 10^{½} = 10^1 = 10$ exponents $\frac{1}{2} + \frac{1}{2} = 1$

but wow! $\sqrt{10} \cdot \sqrt{10} = 10$ also gives 10

so $10^{½} = \sqrt{10}$

We state that 10 elevated to ½ is the same as the square root of 10.
So, instead of writing $\sqrt{10}$, we might as well write $10^{½}$.

As mentioned, this is often an advantage.

5.

And now the tricky one:

$10^0 = 1$

We can see from the exponents in the fraction that:

$\frac{10^1}{10^1} = 10^{1-1} = 10^0 = 1$

Exponent in the numerator: 1

Exponent in the denominator: 1 moved up -1

Calculation of the exponents: 1-1 = 0

So, 10^0 must equal 1.

6.

$(10^2)^3 = 10^2 \cdot 10^2 \cdot 10^2 = 10^6 = 10^{2\cdot 3}$

7.

$(2 \cdot 3)^4 = 6^4 = 1296$

$2^4 \cdot 3^4 = 16 \cdot 81 = 1296$

so

$(2 \cdot 3)^4 = 2^4 \cdot 3^4$

8.

$$\left(\frac{2}{3}\right)^4 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3 \cdot 3} = \frac{16}{81}$$

$$\frac{2^4}{3^4} = \frac{16}{81}$$

so

$$\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$$

Or in letters:

9.

$a^2 \cdot a^3 = a \cdot a \cdot a \cdot a \cdot a = a^5$

10.

$$\frac{1}{b^2 \cdot b^3} = \frac{1}{b^5} = \frac{b^{-5}}{1} = b^{-5}$$

11.

$$\frac{x^4 \cdot x^{-3} \cdot x^5}{x^2 \cdot x^3} = x^1 = x$$

12.

$$y^{½} \cdot y^{½} = y^1 = y$$

13.

$$a^0 = \frac{a^1}{a^1} = 1 \qquad x^0 = \frac{x^1}{x^1} = 1 \qquad 1764^0 = 1$$

Or the long way

$$a^0 = a^{(1-1)} = a^1 \cdot a^{-1} = \frac{a^1}{a^1} = 1$$

Or for fun

$$a^0 = a^{(2-2)} = a^2 \cdot a^{-2} = \frac{a^2}{a^2} = 1$$

$$a^0 = a^{(x-x)} = a^x \cdot a^{-x} = \frac{a^x}{a^x} = 1$$

14.

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^6 = a^{2 \cdot 3}$$

$(a^r)^s = a^{r \cdot s}$

15.

$(a \cdot b)^4 = a^4 \cdot b^4$

$(a \cdot b)^r = a^r \cdot b^r$

16.

$\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$

$\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$

17.

The third root of 8 as an exponentiation: $\sqrt[3]{8} = 8^{\frac{1}{3}} = 2$

which is seen from $8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} = 8^1 = 8$

and the fourth root of 16: $\sqrt[4]{16} = 16^{\frac{1}{4}} = 2$

which is seen from $16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} = 16^1 = 16$

Equations

An equation expresses that what is to the left of the equality sign (=) equals what is to the right of the equality sign. If that is fulfilled the equation is true.

We need equations to find one or more unknown quantities. That happens a lot.

For instance, my salary depends on the number of hours I work. We can state that in an equation:

Salary = hourly rate times number of working hours

or from physics, Newton's Second Law:

force = mass times acceleration

with symbols

$F = m \cdot a$

If we know the numbers for the mass and the acceleration we can multiply them and find the force.

Equations are needed everywhere.

Let us start out with just one unknown, which in mathematics is often called x, and some known numbers. For instance

$x + 3 = 5$

it is easy to see that $x = 2$.

If we have x^1 (which equals x), we talk about a first degree equation.

If we have x^2, we talk about a second degree equation.

If we have x^3, we talk about a third degree equation.

If we have x^4, we talk about a fourth degree equation.

And so on.

Mostly, we need to solve first degree equations. We also have many second-degree equations, while third degree equations are rare and fourth degree equations are extremely rare.

That corresponds well with the fact, that we only have safe methods to solve first degree and second degree equations. Higher degree equations can only be solved by special methods (about this later) or CAS.

In an equation

- We may multiply by the same number (or letter, or other) on both sides of the equation, except 0.
- We may divide by the same number (or letter, or other) on both sides of the equation, except 0.
- We may add the same number (or letter, or other) on both sides of the equation.
- We may withdraw the same number (or letter, or other) on both sides of the equation.

It is crucial that what stands to the left equals what stands to the right. These four rules ensure that equality is maintained.

Remember that, when we multiply, divide, add, or subtract, - it must to be for the *whole* of left side - and the *whole* of right side.

Examples

1.

$x + 3 = 5$

multiply by 2: $2(x + 3) = 2 \cdot 5$

multiply by a: $a(x + 3) = a \cdot 5$

2.

$x + 3 = 5$

divide by 2: $\frac{x+3}{2} = \frac{5}{2}$

divide by a: $\frac{x+3}{a} = \frac{5}{a}$

3.

$x + 3 = 5$

add 2: $(x + 3) + 2 = 5 + 2$

add a: $(x + 3) + a = 5 + a$

4.

$x + 3 = 5$

subtract 2: $(x + 3) - 2 = 5 - 2$

subtract a: $(x + 3) - a = 5 - a$

Using these four rules, we can solve for x (make it stand on its own). We do so step by step. In order to carry on, we need a new sign:

$\Leftrightarrow$ which means "logical equivalence", or synonymous, or the same as. The double arrow shows, that the equation is valid regardless of calculation forward or backwards.

5.

$x + 3 = 5 \quad \Leftrightarrow$

$(x + 3) - 3 = 5 - 3 \quad \Leftrightarrow$

$x + 3 - 3 = 5 - 3 \quad \Leftrightarrow$

$x = 2$

Here we calculated forward. We can also calculate back:

$x = 2 \quad \Leftrightarrow$

$x + 3 - 3 = 5 - 3 \quad \Leftrightarrow$

$(x + 3) - 3 = 5 - 3 \quad \Leftrightarrow$

$x + 3 = 5$

Let us look at the four calculation rules again in new examples:

6.

$\frac{x}{3} = 2$

Here we can multiply by 3 on both sides

$\frac{x}{3} = 2 \quad \Leftrightarrow$

$3 \cdot \frac{x}{3} = 3 \cdot 2 \quad \Leftrightarrow$

$x = 6$

But we might as well say that 3, which is in the denominator on the left side, may be moved to the numerator on the right side

$\frac{x}{3} = 2 \qquad \Leftrightarrow$

$x = 3 \cdot 2 \qquad \Leftrightarrow$

$x = 6$

which is quicker.

7.

$3 \cdot x = 6$

we divide by 3 on both sides

$3 \cdot x = 6 \qquad \Leftrightarrow$

$\frac{3x}{3} = \frac{6}{3} \qquad \Leftrightarrow$

$x = 2$

But we might as well say that 3, which is in the numerator on the left side, may be moved to the denominator on the right side

$3 \cdot x = 6 \qquad \Leftrightarrow$

$x = \frac{6}{3} \qquad \Leftrightarrow$

$x = 2$

which is quicker.

8.

$x - 3 = 5$

we can add 3 on both sides

$x - 3 = 5 \quad \Leftrightarrow$

$x - 3 + 3 = 5 + 3 \quad \Leftrightarrow$

$x = 8$

But we might as well say that 3, which is with a minus on the left side, may be moved to be with a plus on the right side

$x - 3 = 5 \quad \Leftrightarrow$

$x = 5 + 3 \quad \Leftrightarrow$

$x = 8$

which is quicker.

9.

$x + 3 = 5$

we can subtract 3 on both sides

$x + 3 = 5 \quad \Leftrightarrow$

$(x + 3) - 3 = 5 - 3 \quad \Leftrightarrow$

$x = 2$

But we might as well say that 3, which is with a plus on the left side, may be moved to be with a minus on the right side

$x + 3 = 5 \quad \Leftrightarrow$

$x = 5 - 3 \quad \Leftrightarrow$

$x = 2$

which is quicker.

10.

$\frac{2}{x} = 4$ and $x \neq 0$

Here we must add that x cannot be 0- since we cannot divide by zero. The sign $\neq$ means ”not equal to” or ”different to”.

It is seldom mentioned in the text of the problem, that $x \neq 0$, so we have to find out ourselves. It is particularly important, if our further calculation yields x equal to 0. That, cannot be used, and there will be “no solution”.

Here, however, there is no problem

$2 = 4 \cdot x$ $\Leftrightarrow$

$x = \frac{2}{4}$ $\Leftrightarrow$

$x = \frac{1}{2}$

11.

Finally an equation with many operations

$\frac{x}{3} - 2x + 4 - \frac{2}{3} = 6 + \frac{6}{5} - x$

we collect x on the left side, and numbers on the right side

$\frac{x}{3} - 2x + x = 6 + \frac{6}{5} - 4 + \frac{2}{3}$

reduction

$$\frac{x}{3} - x = 2 + \frac{6}{5} + \frac{2}{3}$$

find common denominators

$$\frac{x}{3} - \frac{3x}{3} = \frac{15 \cdot 2}{15} + \frac{3 \cdot 6}{3 \cdot 5} + \frac{5 \cdot 2}{5 \cdot 3} \qquad \Leftrightarrow$$

$$\frac{x-3x}{3} = \frac{30+18+10}{15} \qquad \Leftrightarrow$$

$$\frac{-2x}{3} = \frac{58}{15} \qquad \Leftrightarrow$$

$$15 \cdot (-2x) = 3 \cdot 58 \qquad \Leftrightarrow$$

$$-30x = 174 \qquad \Leftrightarrow$$

$$x = \frac{174}{-30} \qquad \Leftrightarrow$$

$x = -\frac{87}{15}$ which is the precise answer. Or

$x = -5{,}8$ as a decimal number, which here adds up.

Second degree equations

When the unknown or variable (here called x) is squared, we have a second degree equation. For instance

$x^2 - x = 0$

or

$3x^2 - x = -3x + 1$

If we rearrange like this

$3x^2 + 2x - 1 = 0$

and use letters instead of numbers

$ax^2 + bx + c = 0$

we can solve for x using this formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sometimes we may find a quicker way, but this formula always works.

Examples

With our figures the calculation is

$3x^2 + 2x - 1 = 0 \quad \Leftrightarrow$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3} \quad \Leftrightarrow$$

$$x = \frac{-2 \pm \sqrt{16}}{6} \quad \Leftrightarrow$$

$$x = \frac{-2 \pm 4}{6} \quad \Leftrightarrow$$

$x = \frac{2}{6} = \frac{1}{3}$ and $x = \frac{-6}{6} = -1$

yielding two solutions: $\frac{1}{3}$ and -1

That is new, but we can insert the two values in the original equation to find that it is true.

First we insert $\frac{1}{3}$ which renders

$3(\frac{1}{3})^2 + 2(\frac{1}{3}) - 1 = 0 \quad \Leftrightarrow$

$3(\frac{1}{9}) + 2(\frac{1}{3}) - \frac{3}{3} = 0 \quad \Leftrightarrow$

$\frac{3}{9} + \frac{2}{3} - \frac{3}{3} = 0 \quad \Leftrightarrow$

$\frac{1}{3} + \frac{2}{3} - \frac{3}{3} = 0 \quad \Leftrightarrow$

$0 = 0$

which is true

Then we insert -1 which renders

$3(-1)^2 + 2(-1) - 1 = 0 \quad \Leftrightarrow$

$3 \cdot 1 + 2(-1) - 1 = 0 \quad \Leftrightarrow$

$3 - 2 - 1 = 0 \quad \Leftrightarrow$

$0 = 0$

and that is true also. So yes, we have two valid answers, to values of x, two solutions. We also say that the equation has two “roots”.

If we for instance found

$2 \ = \ 0$

it clearly is ”false”, and thus the number we tested is not a root, - meaning: it does not fulfil the equation.

Proof

Most people just use the solution formula, but here is presented that it is correct. Actually, the proof is quite complicated:

We have arranged the second degrees equation, so that it looks this way

$$ax^2 + bx + c \ = \ 0 \qquad \Leftrightarrow$$

we multiply by 4a on either side

$$4a\,(ax^2 + bx + c) \ = \ 0 \qquad \Leftrightarrow$$

multiply 4a into the parenthesis

$$4a^2x^2 + 4abx + 4ac \ = \ 0 \qquad \Leftrightarrow$$

add b^2 on either side

$$4a^2x^2 + 4abx + 4ac + b^2 \ = \ b^2 \qquad \Leftrightarrow$$

move 4ac

$$4a^2x^2 + 4abx + b^2 \ = \ b^2 - 4ac \qquad \Leftrightarrow$$

Use one of the remarkable identities (square rules) on the left side

$$(2ax + b)^2 \ = \ b^2 - 4ac \qquad \Leftrightarrow$$

calculate the square root on either side.

Here, however, we need to interfere: If we look at the left side $(2ax + b)$ may originate from a negative number, which becomes positive when squared (minus times minus gives plus). That cannot be seen, so we have to say that (2ax + b) may be negative or positive. We write ± and move it to the right side of the equation. That is allowed since the left side equals the right side:

2ax + b = $\pm\sqrt{b^2 - 4ac}$ ⇔

move b

2ax = -b ± $\sqrt{b^2 - 4ac}$ ⇔

divide by 2a on either side

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

and the formula is proved.

More theory

If we look at the formula again

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

we can focus on the square rooted entity

$b^2 - 4ac$

and call it the *discriminant*, d

d = $b^2 - 4ac$

the discriminant means ”the one making the difference”. What difference? The difference that shows if the second degrees equation has got two, or one, or no solution:

d = positive	two solutions (two roots)
d = 0	one solution (one root)

$d = \text{negative}$ no solution (no root)

we cannot calculate the square root of a negative number.

”no solution” actually has a geometric meaning, which we shall see later in the chapter about the parabola.

Example

If the second degree equation does not have a constant part (no c), we may use the zero solution:

$4x^2 - x = 0 \quad \Leftrightarrow \quad x(4x - 1) = 0$

where either x has to be 0, or the parenthesis has to be 0. =>

$x = 0$ or $x = \frac{1}{4}$

Higher degree equations

For first degree equations, we may have up to one solution.

For second degree equations, we may have up to two solutions.

For third degree equations, we may have up to three solutions.

For fourth degree equations, we may have up to four solutions.

and so on.

In first degree equations, we solve for x by making it stand on its own on the left side of the equation.

In second degree equations we arrange and use the formula to find x.

In third degree equations, and higher, we guess a solution, insert it in the equation, and see if it is correct. It may seem strange, that we may guess, but that is fine in mathematics, if the guessed value(s) is tested.

Often we will use CAS. The electronics in CAS do as we do. It makes a guess and tests it, checks how big the error is, make a new and closer guess, and so on. The method is called *iteration* (repetition). CAS does that quickly, but in general we can also get close to the result after 3-4 guesses.

Examples

$x^3 = 27$

we guess $x = 3$ and make a test

$3^3 = 3 \cdot 3 \cdot 3 = 27$

Which is true. So - 3 is one solution.

What about -3. We make a test

$(-3)^3 = -3 \cdot -3 \cdot -3 = -27$

Which is false. -3 is no solution.

We cannot find other roots. Numerical numbers bigger than 3 can never be a solution, and it is easy to test numbers between -3 and 3. So, in this case we are sure that there are no other roots than 3.

But what about:

$x^3 - x^2 = 0 \quad \Leftrightarrow$

Maybe it becomes more clear if we write

$x^3 = x^2$

We guess 0, yes. We guess 3, no. We guess 2, no. We guess 1, yes. Maybe -1:

$(-1)^3 = 1^3$

$-1 = 1$

which is false, so, no.

Theoretically, three roots are possible, but it is easy to see that roots bigger than numeric 1 ($|1|$) are not possible. Also, it is easy to see that roots smaller than numeric 1 ($|1|$) are not possible. Only $x = 0$ and $x = 1$ are roots.

Two equations with two unknowns

If we have two unknowns, we need two different equations.

If we have three unknowns, we need three different equations.

And so on.

There are two methods of solving two equations with two unknowns.

The most logic method is to isolate in one equation and insert into the other. This method is widely used for all subjects.

The quickest method in simple cases is the equal coefficients method.

Examples

Let us pretend that we are in a laboratory and measure something in two different ways giving us two different equations. Here we call the unknowns x and y - but they may stand for pressure and temperature, or time and number of bacteria, or something else. And we pretend that we have found these two relations:

$x + y = 4$ and $2x = 2y + 2$

1.

$x + y = 4$ and $2x = 2y + 2$

we isolate x in the first equation and insert into the other

$x = 4 - y$ inserted $2(4 - y) = 2y + 2$

on with equation 2

$2(4 - y) = 2y + 2 \quad \Leftrightarrow$

$8 - 2y = 2y + 2 \quad \Leftrightarrow$

$6 = 4y \quad \Leftrightarrow$

$y = \frac{6}{4} = \frac{3}{2}$

which we insert into one of the original equations, no matter which. We choose to insert into equation 1:

$x + \frac{3}{2} = 4 \quad \Leftrightarrow$

$x + \frac{3}{2} = \frac{8}{2}$

$x = \frac{5}{2}$

complete answer $x = \frac{5}{2}$ and $y = \frac{3}{2}$.

2.

Same problem solved with the equal coefficients method:

$x + y = 4$

$2x = 2y + 2$

Here it is easier if we write x above x and y above y:

$x + y = 4$

$2x - 2y = 2$

Now we choose equal coefficients before x so we multiply equation one by 2:

$2x + 2y = 8$

$2x - 2y = 2$

Then we say equation 1 minus equation 2.

2x - 2x gives zero, 2y - (-2y) gives 4y, 8 - 2 gives 6:

$4y = 6 \quad\quad\quad\quad \Leftrightarrow$

$y = \frac{6}{4} = \frac{3}{2}$

which is inserted into one of the original equations. We choose equation 2:

$2x = 2 \cdot \frac{3}{2} + 2 \quad\quad\quad\quad \Leftrightarrow$

$2x = 5 \quad\quad\quad\quad \Leftrightarrow$

$x = \frac{5}{2}$

complete answer $x = \frac{5}{2}$ and $y = \frac{3}{2}$

The same as before, of course.

Why is it allowed to subtract one equation from the other? Because we may subtract the same entity on either side. On the left side we subtract (2x - 2y), and must do the same on the right side, only we choose to subtract what (2x - 2y) is equal to, namely 2.

Functions and proportionality

A function is a technical term used when something depends on something else. A function is written as an equation and the function flow may be shown in a diagram. More about this in Part 2.

For instance, my salary depends on how many hours I work. We can write it in an equation:

salary = hourly rate times number of working hours

In other words, my salary is a *function* of my hourly rate and how many hours I work.

Or from Physics, Newton's Second Law:

Force = mass times acceleration

with symbols

$F = m \cdot a$

The force depends on the mass and the acceleration. Or: The force is a function of the mass and the acceleration.

Let us again look at the function/law of nature, Newton's Second Law:

$$F = m \cdot a \quad (1) \qquad \Leftrightarrow \qquad a = F \cdot \frac{1}{m} \quad (2)$$

Expression (1) shows that if the mass is twice as big, the force will be twice as big. We state that F and m are directly proportional. Furthermore, F and a are also directly proportional.

(So, if both m and a are doubled, F will be four times as big).

Expression (2) shows that if the mass is doubled, the acceleration will be half. We state that a and m are inversely proportional.

Intervals and inequalities

We need a long working table which must be longer than 3 meters and shorter than 4 meters. If it is precisely 3 meters, it is slightly too short, - and if it is precisely 4 meters, it is slightly too long. We need a table in the interval

]3;4[(we know it is in meters, but we do not write it).

3 is not included, neither is 4. An open interval.

If 3 and 4 meters are usable, we can use a table in the interval

[3;4]

3 is included, so is 4. A closed interval.

If 3 meters is usable but 4 meters are slightly too long

[3;4[

3 is included, but 4 is not included. A half-open (semi-open) interval.

It can also be written as a double inequality:

$3 < \text{length} < 4$ which corresponds to the interval]3;4[

$3 \leq \text{length} \leq 4$ which corresponds to the interval [3;4]

$3 \leq \text{length} < 4$ which corresponds to the interval [3;4[

or shown in a small figure

○————○ the open interval

•————• the closed interval

•————○ the half-open interval

3 4

Examples

The easiest way is to write an interval or make a small sketch.

Inequalities are not as common, but let us see what the signs mean:

$<$ means smaller than

$\leq$ means smaller or equal to

$>$ means bigger than

$\geq$ means bigger than or equal to

The small part is placed by the point of the sign. The big part is placed at the "mouth" of the sign.

Instead of writing

$3 < \text{length} < 4$

We may write

$4 > \text{length} > 3$

If x is length and the limits are a and b, we write

$a < x < b$ a double inequality

which we rarely use for further calculation. It is more comfortable to split into two single inequalities

$a < x$ and $x < b$

and make calculations on each. We may:

add the same on either side

subtract the same on either side

multiply by the same positive number on either side

divide by the same positive number on either side

If we will multiply or divide by the same negative number on either side, we must turn around the inequality sign because we turn things "upside-down". An example:

$a < x$ a is small and x is big

if we multiply by for instance -2 on either side, we must turn the sign:

$-2a > (-2)x$

Because now $-2a$ is big and $-2x$ small.

If we return to the example with the table and only consider the low limit:

$3 < \text{length}$

It makes no sense to add for instance 2 on either side:

$3 + 2 < \text{length} + 2$

However, it is allowed as a calculation tool, and later we will see it used that way.

Imaginary numbers, briefly

Imaginary numbers are not real but have to be imagined.

What is meant?

$\sqrt{64} \;=\; 8$ is well known

$\sqrt{-64}$ cannot be done, but if we change it a bit:

$\sqrt{-64} \;=\; \sqrt{(-1) \cdot 64} \;=\; \sqrt{(-1)} \;\cdot\; \sqrt{64}$ that is quite ok

$\sqrt{(-1)}$ we name I, that is also allowed, and then we have

$\mathrm{I} \cdot \sqrt{64} \;=\; \mathrm{I} \cdot 8$

So $\sqrt{-64} \;=\; \mathrm{I} \cdot 8$

which enables us to continue as if nothing has happened; only now, we are in the world of imaginary numbers. And suddenly we for instance have an imaginary solution to a second degree equation which had ”no solution”.

CAS obtains both real and imaginary solutions. Most CAS (typically calculators) are programmed to give real answers only. However, some programs also give an imaginary answer, for instance I · 8. That may be avoided by asking for ”Real Domain” or the like.

Example

Let us find the imaginary roots to a second degrees equation that has no real roots:

$x^2 + 2x + 5 \;=\; 0 \qquad\qquad \Leftrightarrow$

$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$ ⇔ negative discriminant, with no real roots

$x = \frac{-2 \pm \sqrt{-16}}{2}$ ⇔

$x = \frac{-2 \pm 4\sqrt{-1}}{2}$ ⇔

x = $-1 + 2\sqrt{-1}$ and x = $-1 - 2\sqrt{-1}$ or

x = $-1 + 2$ I and x = $-1 - 2$ I

thus two imaginary roots.

The combined use of real numbers and imaginary numbers is called complex numbers. Complex numbers may serve as a mathematical tool and will be described again at the end of this book.

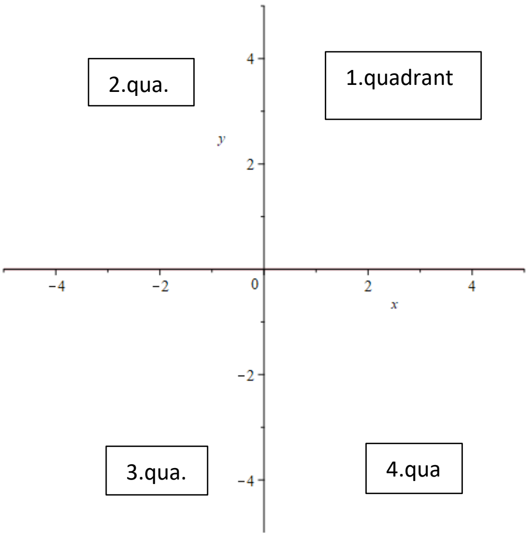
Part 2. The coordinate system in the plane (2D) and functions

The coordinate system and distance

We live in a world of three dimensions, we call it the space and it consists of length, width, and height.

If we work in two dimensions, we call it the plane, and it consists of two directions for instance horizontal and vertical. We may also call the directions for axis. Then we have the *first-axis* and the *second-axis*; or in more technical terms: The *abscissa* and the *ordinate*, both from Latin. Abscissa means ”out (ab) from here (cis)”, which may be pictured by standing at the starting point and looking horizontally at the horizon. The ordinate means the ordinary, which is vertical (all other directions would not be ordinary).

In mathematics we often use the words x-axis and y-axis,



but they can be called other things. In physics the first axis could be t for time, and the second axis could be v for velocity (velox in

Latin). In economy the first axis may be months and the second axis may be costs. And so on.

The axis divide the plane in four quarters named the four quadrants. The first quadrant is where x and y are positive (both are +). Then we rotate counter clockwise to the 2. 3. and 4. quadrant.

The axis form a right angle and intersect in a common starting point, denoted like this: $(x,y) = (0,0)$. The starting point is called Origo (ancient Greek) or just O.

At the axis we chose a scale suitable for the task. Usually, we chose the same scale for the two axes, but that depends on what we are going to plot. If the scales are alike, we use the technical term: equidistant.

In all it is called a coordinate system (co(with) ordinate(the ordinary) system). It is being used everywhere.

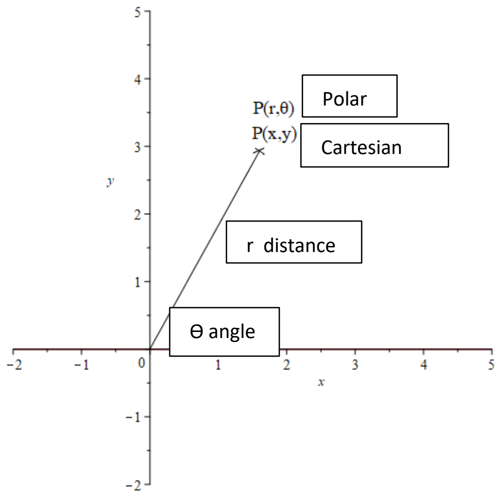
For instance, the coordinate system is used to show how a function varies: the straight line function, the parabola function, the sinus-function, and so on.

We consider x first, and y as what follows. Therefore, the x-values of a function are also called the *domain*, and the y-values are called the *range* (sometimes: the amount of value). Denotations are not commonly used because the words domain and range are brief enough and very informative, but if we call the function, f, the denotations are: Domain, D(f) - and Range, G(f). (R(f) for range would have been the logical choice, but R is used for something else).

The demand for a function is that for each x-value there is only one y-value. Therefore a function flow in a coordinate system cannot go forth and back since that would imply more y-values for one x-value. If that is needed we talk about a vector function or a parameter function which will be discussed in Part 4.

The ordinary rectangular coordinate system is also called the Cartesian coordinate system after the mathematician Descartes.

Coordinates may also be denoted by polar coordinates: (distance from Origo , angle with +x axis). See the figure:



We will consider polar coordinates a little more at the end of the book.

Now it is about normal (Cartesian) coordinates.

Distance

Below is shown a coordinate system with three points called A, B and C with the coordinates: A(2,3) B(5,4) C(1,-3).

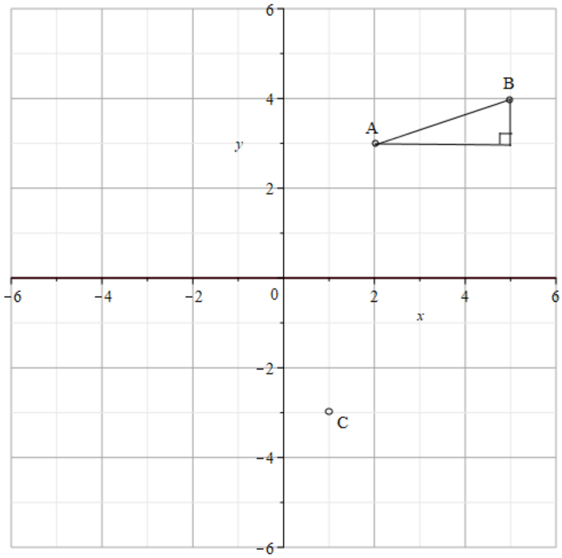
A and B are in the first quadrant, while C is in the fourth quadrant.

We denote the distance between A and B, d, and find it by using one of the oldest and most important formulas, Pythagoras, which is valid for rectangular (90^o) triangles. We sketch a helping triangle and see that the side in the x-direction (horizontal) has the length 3, while the side in the y-direction has the length 1.

In other words, we found the lengths by saying

x value for B minus x value for A $= x_B - x_A = 5 - 2 = 3$

y value for B minus y value for A $= y_B - y_A = 4 - 3 = 1$



Then Pythagoras states:

$d^2 = 3^2 + 1^2 \quad \Leftrightarrow$

$d = \sqrt{3 \cdot 3 + 1 \cdot 1} \quad \Leftrightarrow$

$d = \sqrt{10}$ thus, the distance is $\sqrt{10}$

We can also denote the distance from A to B as $|AB|$. The straight parenthesis means:

length = numerical value = magnitude of the number

In letters we obtain the *distance formula*:

$|AB|^2 = (x_B - x_A)^2 + (y_B - y_A)^2$

We may also compute the square root on both sides - some tables do.

The distance formula is just Pythagoras in another way.

Examples

1.

Above we found $|AB| = \sqrt{10}$ by saying B minus A.

We get the same distance if we find $|BA|$ by saying A minus B:

$|BA|^2 = (x_A - x_B)^2 + (y_A - y_B)^2 \quad \Rightarrow$

$|BA|^2 = (2 - 5)^2 + (3 - 4)^2 \quad \Leftrightarrow$

$|BA|^2 = (-3)^2 + (-1)^2 \quad \Leftrightarrow$

$|BA|^2 = 9 + 1 \quad \Leftrightarrow$

$|BA| = \sqrt{10}$

Surely, the distance from A to B is the same as the distance from B to A.

2.

We will also find the distance from C to A.

To keep things in order we say ”end minus start”. For $|CA|$ that is A minus C.

$|CA|^2 = (x_A - x_C)^2 + (y_A - y_C)^2 \quad \Rightarrow$

$|CA|^2 = (2 - 1)^2 + (3 - (-3))^2 \quad \Leftrightarrow$

$|CA|^2 = (1)^2 + (6)^2 \quad \Leftrightarrow$

$|CA|^2 \;=\; 1 + 36 \qquad\qquad \Leftrightarrow$

$|CA| \;=\; \sqrt{37}$

For |AC| it is C minus A:

$|AC|^2 \;=\; (x_C - x_A)^2 + (y_C - y_A)^2 \qquad => $

$|AC|^2 \;=\; (1 - 2)^2 + (-3 - 3)^2 \qquad \Leftrightarrow$

$|CA|^2 \;=\; (-1)^2 + (-6)^2 \qquad \Leftrightarrow$

$|CA|^2 \;=\; 1 + 36 \qquad \Leftrightarrow$

$|CA| \;=\; \sqrt{37}$

Same answer.

The sign => means *logical consequence*. We use it when we can only go forward in the calculation, but not back. That happens when we introduce something from outside. Here we introduce our numbers in the formula.

Now we will consider important functions, their equations and their curves in coordinate systems (diagrams).

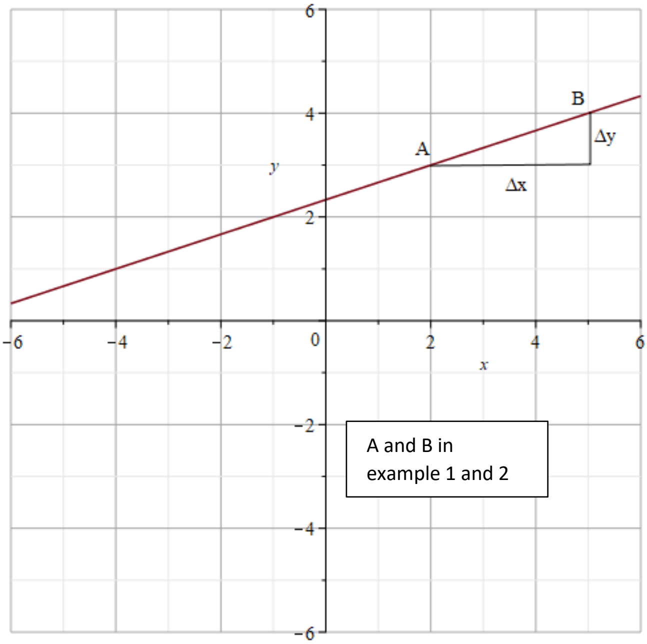
We put many efforts into the straight line, because a lot within physics, biology, economy, design, etc., is described and explained by straight lines. Furthermore, we lay down the foundation for considering other functions.

The straight line (The linear function)

We write the equation for the linear function in two ways:

1.

Let us use the figure from the former chapter and draw a straight line through the points A and B:



Also, we will use the helping triangle again: Using letters the side in the x-direction (horizontal) has the length:

$x_B - x_A = \Delta x$

and the side in the y-direction (vertical) has the length:

$y_B - y_A = \Delta y$

We use the Greek letter Δ (delta) when we describe a change or difference. Here it is the difference in x-values and y-values of the points. End minus start. (In physics Δt may be a temperature difference, in economy ΔI may be a difference in income, etc.). The technical word for Δ is ”change or difference” which may also be negative.

Now we are interested in a number that tells us about the slope of a line. The line has a big slope if Δy is big compared with Δx. The line has a small slope if Δy is small compared with Δx.

This is shown by the fraction $\frac{\Delta y}{\Delta x}$ which is called the slope and usually denoted a.

So, we define:

$$\text{slope of a line} = a = \frac{\Delta y}{\Delta x} = \frac{\text{difference in } y}{\text{difference in } x}$$

We go on calculating:

$$a = \frac{\Delta y}{\Delta x} \quad \Leftrightarrow \quad a \cdot \Delta x = \Delta y \quad \Leftrightarrow \quad \Delta y = a \cdot \Delta x \quad \Leftrightarrow$$

$$y_B - y_A = a \cdot (x_B - x_A) \quad \Leftrightarrow$$

$$y_B = a \cdot (x_B - x_A) + y_A$$

which is the equation for our straight line. Now, we named the points A and B. They may have other names, and many tables state:

$$y = a \cdot (x - x_1) + y_1$$

which is the equation for almost all straight lines (except for vertical lines).

a is the slope, and (x_1 , y_1) is a known point on the line. If we know these, we can write the equation for a certain line and sketch it in a coordinate system.

That was method *1*.

2.

And now method *2*, in which we continue calculations using the already derived equation for the straight line:

$y = a \cdot (x - x_1) + y_1$

where (x_1, y_1) is a known point on the line. Let us choose point (0, b) which is on the y-axis with $y = b$. This point is inserted and we have

$y = ax + b$

which is the most common equation for almost all straight lines (except the vertical lines).

a is the slope and b is where the line intersects the y-axis. If we know a and b, we can write the equation for a certain line, and we can sketch it in a coordinate system.

Examples

1.

We will find the equation for the line through points A(2,3) and B(5,4) using method *1*:

$y = a \cdot (x - x_1) + y_1$

We must find a, x_1 and x_2

$$a = \frac{\Delta y}{\Delta x} = \frac{4-3}{5-2} = \frac{1}{3}$$

We chose to insert the coordinates of point A: $(x_1, y_1) = (2,3)$

$y = \frac{1}{3}(x - 2) + 3 \quad \Leftrightarrow \qquad$ reduction

$$y = \frac{1}{3}x - \frac{2}{3} + 3 \quad \Leftrightarrow$$

$$y = \frac{1}{3}x - \frac{2}{3} + \frac{9}{3} \quad \Leftrightarrow$$

$$y = \frac{1}{3}x + \frac{7}{3}$$

We might as well have inserted the coordinates of point B(5,4). Point B is also on the line, and will therefore fulfil the equation just as well:

$$y = \frac{1}{3}(x - 5) + 4 \quad \Leftrightarrow$$

$$y = \frac{1}{3}x + \frac{7}{3}$$

same answer of course. If we knew the coordinates of other points on the line, it would render the same equation.

What we found is the equation for "our" line.

2.

We will now find the equation for the line through points A(2,3) and B(5,4) using method *2*:

$$y = a \cdot x + b$$

a is

$a = \frac{\Delta y}{\Delta x} = \frac{4-3}{5-2} = \frac{1}{3}$

and b is found by inserting A or B (same result) into the equation. We chose A:

$3 = \frac{1}{3} \cdot 2 + b \qquad \Leftrightarrow$

$b = \frac{7}{3}$

a and b inserted into the equation

$y = \frac{1}{3} x + \frac{7}{3}$

gives us the equation for "our" line. Same answer.

Also, we will see what y becomes for $x = 3$

$y = \frac{1}{3} 3 + \frac{7}{3} = \frac{10}{3}$

and for $x = 17$

$y = \frac{1}{3} 17 + \frac{7}{3} = 8$

Now we know that $(3, \frac{10}{3})$ and $(17,8)$ are points on our line.

3.

Another example using method *2* - this time without numbers:

My salary (y) equals my hourly rate (a) times how many hours (x) I work (a and x are directly proportional).

salary = hourly rate · number of working hours =>

$y = a \cdot x$

If I also get a fixed amount regardless of how many hours I work, it can be written:

salary = hourly rate · number of working hours + fixed amount $\Rightarrow$

$y = a \cdot x + b$

where b is the fixed amount.

4.

Two lines are called l and m. Do they intersect? If so, at which coordinates?

l: $y = -x + 3$

m: $y = 2x$

First we note that their slope is different - so we know that they will intersect somewhere. If the slopes were similar, they would be parallel and would never intersect.

This is two equations with two unknowns:

y from l is inserted into m

$-x + 3 = 2x \quad \Leftrightarrow$

$-x - 2x = -3 \quad \Leftrightarrow$

$x = \frac{-3}{-3} = 1$

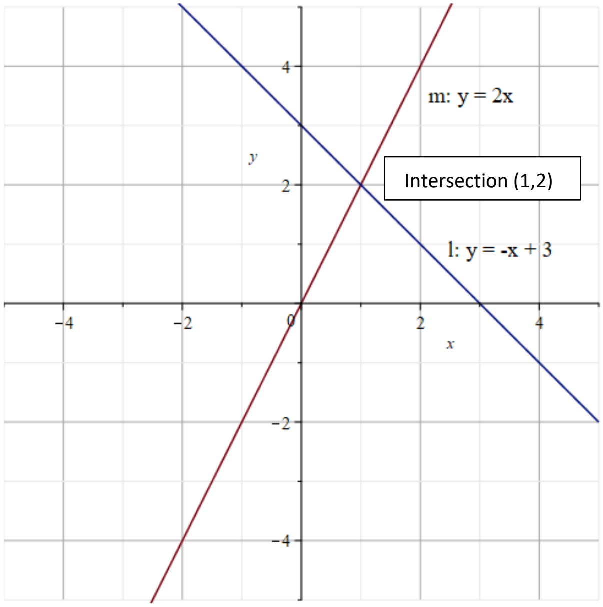
which is inserted in one of the old equations, here m

$y = 2 \cdot 1 \quad \Leftrightarrow$

$y = 2$

Thus, intersection in (1, 2)

Shown in a diagram (another word for a coordinate system):



l is decreasing (the slope is negative).

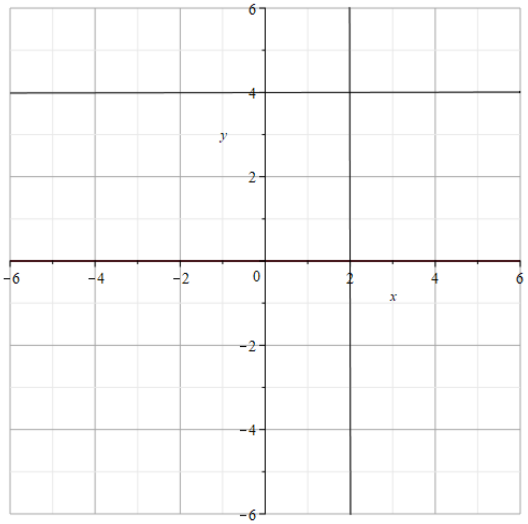
m is increasing (the slope is positive).

The point of intersect is read as (1, 2). This is called a graphical solution, and it corresponds with the calculation.

More theory

A line continues endlessly in both directions. A line segment goes from one point to another point. For instance, the line segment from A to B as shown in a later chapter: Distance.

There are four special straight lines:



- The horizontal lines with slope 0. Here is shown the line with the equation $y = 4$
  Another horizontal line is the x-axis itself, with the equation $y = 0$.
- The vertical lines with slope ∞ (a sign that means infinite). Consequently, they are *not* determined with the usual equations. Here is shown the line with the equation $x = 2$
  Another vertical line is the y-axis itself, with the equation $x = 0$.

By the way, it may be read that the lines $y = 4$ and $x = 2$ intersect at point (2,4)

Except for the special lines just mentioned, the slopes of two right angled (= orthogonal) lines multiplied will yield -1. If the lines are called l and m, the following applies:

$a_l \cdot a_m = -1$

Therefore, we can see if two lines are orthogonal by multiplying their slopes and see if it yields -1. For instance, we can check two lines in a building to see if the corner of some walls have an angle of 90°.

Proof

The diagram shows two orthogonal lines called n and m. The equations are shown as well.

n can pivot 90° round the point of intersection and become m, and the helping triangle follows.

n's slope reads: $a_n = \frac{2}{3}$

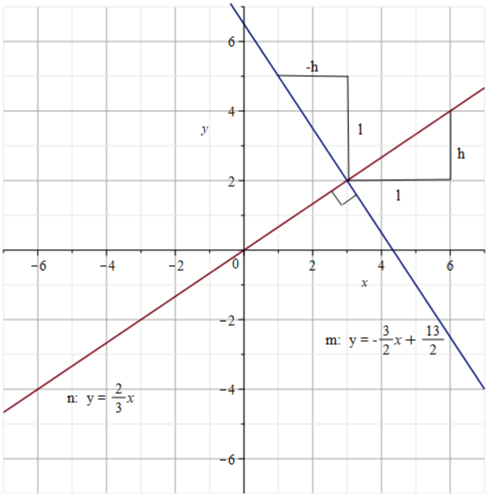
m's slope reads: $a_m = \frac{3}{-2} = -\frac{3}{2}$

multiplied it yields:

$a_n \cdot a_m = \frac{2}{3} \cdot \left(\frac{3}{-2}\right) = -1$ shown with numbers

or in letters:

$a_n \cdot a_m = \frac{h}{l} \cdot \left(\frac{l}{-h}\right) = -1$ proven with letters



The parabola

The technical term for a parabola is a ”second degree polynomial”. Second degree because x is raised to the power of 2. ”Poly” means several and ”nomial” means part.

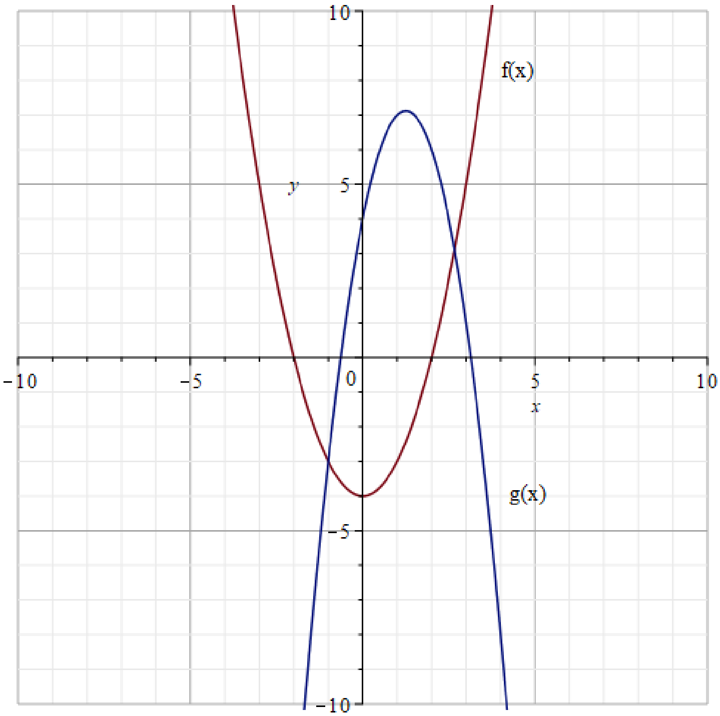
We have nicknames for the figures we use, and this beautiful curve is named parabola, which means comparison, maybe because the figure is symmetric when we compare the two halves.

The diagram shows two parabolas. One with the equation

$$f(x) = x^2 - 4$$

and another with the equation

$$g(x) = -2x^2 + 5x + 4$$



For both parabolas, y is a function of x (y depends on x), but they cannot both be called y so the first we call f(x) (we say f of x), and the other is called g(x).

The equation for all parabolas is:

$y = ax^2 + bx + c$ or

$p(x) = ax^2 + bx + c$ p for polynomial.

A big a means that the parabola is narrow. A small a means that the parabola is wide.

A positive a means that the branches face up. A negative a means that the branches face down.

b moves the parabola in the x and y direction, while c only moves it in the y direction.

When we solved second degree equations, we had:

$ax^2 + bx + c = 0$

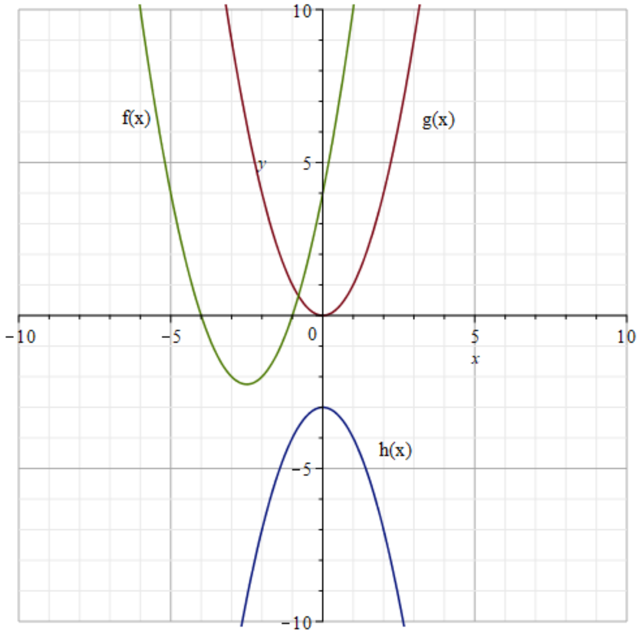
Now vi use the second degree equation for a parabola, and 0 must therefore be because $y = 0$.

y is 0 on the x-axis. Therefore, for a parabola, the solution to a second degree equation must be where the parabola intersects the x-axis.

Often, there are two roots for x in the points where the parabolas two branches intersect the x-axis (like f(x) in the next diagram).

If the discriminant is zero, there is only one root, which is the peak of the parabola touching the x-axis (like g(x) in the diagram).

No solution means that the parabola does no intersect or touch the x-axis (like h(x) in the diagram).



So, when vi use the second degree equation for the geometry of the parabola, ”no solution” does have a meaning.

The Parabola has a vertex (“turning point”) with the coordinates:

$\left(\frac{-b}{2a}, \frac{-d}{4a}\right)$ which we will prove:

A parabolas vertex has but one x-value for one y-value. The vertex for a parabola with vertex on the x-axis, has the y-value of 0. Thus, the discriminant is 0 and the x-value becomes:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ => $x = \frac{-b}{2a}$ for $d = 0$

If we move the parabola up or down, the y-value will change, while the x-value will remain

$x = \frac{-b}{2a}$

which we insert into the equation of the parabola in order to find the y-coordinate:

$y = ax^2 + bx + c$ $\Rightarrow$

$y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$ $\Leftrightarrow$

$y = \frac{abb}{4aa} - \frac{bb}{2a} + c$ $\Leftrightarrow$

$y = \frac{bb}{4a} - \frac{2bb}{4a} + c$

$y = -\frac{bb}{4a} + c$ $\Leftrightarrow$

$y = \frac{-bb+4ac}{4a}$ $\Leftrightarrow$

$y = \frac{-d}{4a}$

x and y combined: Vertex $\left(\frac{-b}{2a}, \frac{-d}{4a}\right)$ and the formula is proved.

Examples

1.

f(x) in the diagram just shown

$f(x) = x^2 + 5x + 4$ or

$y = x^2 + 5x + 4$ here we must remember, that this y-value is for f(x) only

Where does it intersect the x-axis?

That happens where $y = 0$ which is on the x-axis

Therefore, we insert 0 for y and solve the second degree equation

$x^2 + 5x + 4 = 0$

using the formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\Rightarrow$

$x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$ $\Leftrightarrow$

$x = \frac{-5 \pm 3}{2}$ $\Leftrightarrow$

x = -4 and -1

which, corresponds with the diagram.

Where does it intersect the y-axis?

That happens where x = 0 which is on the y-axis

Therefore, we insert 0 for x and solve the second degree equation

$y = 0^2 + 5 \cdot 0 + 4$ $\Leftrightarrow$

y = 4

which, corresponds with the diagram.

It is called factorization, if we want to replace x^2 by two parentheses with x^1. For instance,

$y = x^2 + 5x + 4$ may be factorized to $y = (x + 4)(x + 1)$

or

$y = 2x^2 + 10x + 8$ when factorized $y = 2(x + 4)(x + 1)$

It is done by putting the equations factor, a (here 2), out from the parentheses, find the roots, and form the first parenthesis as: $x - root_1$ and the other as $x - root_2$

Factorization will be proved in the section: Proof of factorization of a second degree polynomium.

2.

h(x) in the diagram just shown has the equation

$h(x) = -x^2 - 3$

Where does it intersect the x-axis?

On the x-axis y is 0. Here: $h(x) = 0$. So

$-x^2 - 3 = 0 \quad \Leftrightarrow$

$x^2 = -3$

which is not possible, therefore: ”no solution”. This means that the parabola h(x) does not intersect the x-axis. It corresponds with what we see in the diagram.

3.

$h(x) = -x^2 - 3$

Where is the vertex?

formula $\left(\frac{-b}{2a}, \frac{-d}{4a} \right)$

where $a = -1 \qquad b = 0$ (there is no x)

and $d = b^2 - 4ac = 0 - 4 \cdot (-1) \cdot (-3) = -12$

which inserted gives

$\left(\frac{0}{2\cdot(-1)}, \frac{-(-12)}{4(-1)} \right) = (0, -3)$

which corresponds with the diagram.

4.

Let us go back to the diagram with the parabolas:

$f(x) = x^2 - 4$

$g(x) = -2x^2 + 5x + 4$

Where do they intersect one another?

They intersect in a point (maybe two points) that fulfil both equations, where the one equation equals the other: =>

$f(x) = g(x)$ =>

$x^2 - 4 = -2x^2 + 5x + 4$

which is a second degrees equation to be arranged and solved:

$3x^2 - 5x - 8 = 0$

formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ =>

here: $x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 3 \cdot (-8)}}{2 \cdot 3}$ ⇔

$x = \frac{5 \pm 11}{6}$ ⇔

$x = \frac{8}{3}$ and -1

We call the x-coordinates of the intersection points

$x_1 = \frac{8}{3}$ and $x_2 = -1$

We read the diagram to find that it corresponds.

We find the y-coordinates by insertion into one of the parabola equations. It does not matter which, because the intersection points are located on both parabolas. We chose f(x):

$$y_1 = x_1^2 - 4 = \left(\frac{8}{3}\right)^2 - 4 = \frac{28}{9}$$

$$y_2 = x_2^2 - 4 = (-1)^2 - 4 = -3$$

Thus, the two intersection points are:

$\left(\frac{8}{3}, \frac{28}{9}\right)$ and $(-1, -3)$

Which also corresponds with the diagram.

Intersection points for *all* curves is found by having:

equation for curve 1 = equation for curve 2

More on the parabola

Many laws of nature are second degree equations that may be shown as parabolas in a diagram. For instance the formula for kinetic energy (motion energy), E_{kin} :

$E_{kin} = \frac{1}{2} \cdot m \cdot v^2$ where m is mass, v is velocity.

In a v, E_{kin} - diagram (v on first axis, and E_{kin} on second axis) we will get half of a parabola. More about this later.

Parabolas for technical usage: If a parabola is rotated around its centre line we will get a parabola dish, which is a 3D figure. Among other things, it is used in front lights in cars where the bulb is located in the focus point of the parabola dish. Light radiated backwards and laterally will hit the parabola and be reflected forwardly. A very “open” parabola dish can also receive, for instance TV-signals, which are reflected to a receiving device at the focus point. ..and much more.

Polynomials

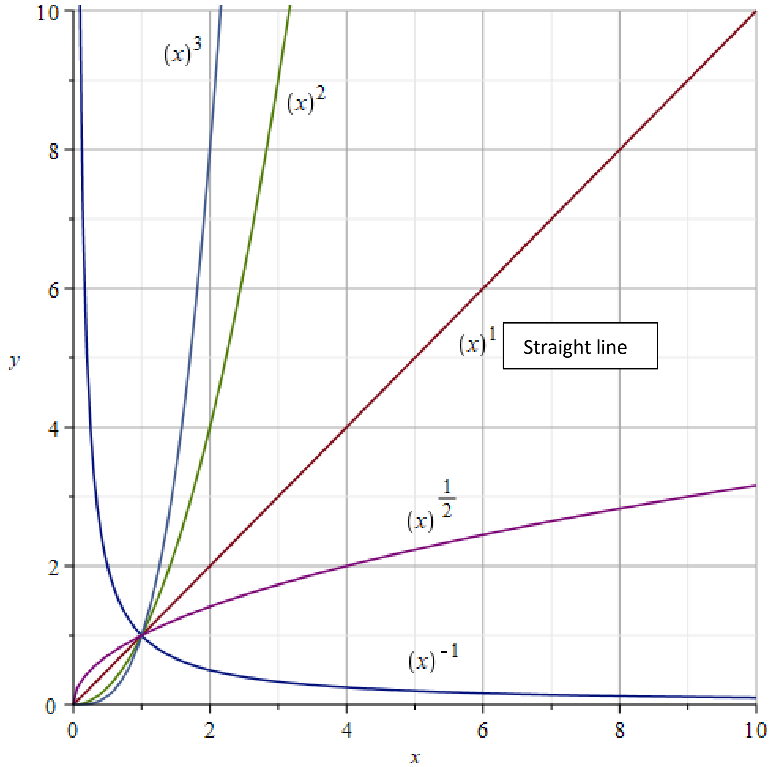
Polynomials have x as basis number and an exponent, n

$y = x^n$ or

$f(x) = x^n$

The recently discussed parabola is a polynomial with exponent 2.

The diagram shows four relevant polynomials in the first quadrant:



x is in a parenthesis, because the sketch program demands it, - mathematics does not demand it.

A straight line $y = x^1 = x$ is shown for comparison.

Examples

1.

$y = x^3$ is called a third degree polynomial.

For instance the volume of a sphere:

$V = \frac{4}{3} \cdot \pi \cdot r^3$ where radius is to the power of three.

2.

$y = x^2$ is called a second degree polynomial or a parabola which was discussed in the latter chapter.

For instance the formula for kinetic energy (motion energy):

$E_{kin} = \frac{1}{2} \cdot m \cdot v^2$ where m is mass, v is velocity.

If the mass is a constant (maybe a known number), E_{kin} will be a function of (depending of) v^2

3.

$y = x^{½} \Leftrightarrow y = \sqrt{x}$ is called the square root function

Again, we may use the formula for kinetic energy as an example, only now we solve for the velocity, v

$v = \left(\frac{2E}{m}\right)^{½}$

If the mass is a constant (maybe a known number), v will be a function of (depending of) $E_{kin}^{½}$

4.

$y = x^{-1} = \frac{1}{x}$ is called the reciprocal function

Reciprocal must not be confused with ”reverse”, which we will discuss later.

For instance Boyle Mariotte’s law from physics. It is valid (within limits) for gases and says that pressure times volume is a constant (a certain number) if temperature is kept constant:

$p \cdot V = k \quad \Leftrightarrow$

$p = k \cdot V^{-1}$

So, in a V,p diagram we have a curve similar to the one in the x,y diagram denoted $(x)^{-1}$. The curve is called a hyperbole, which is ancient Greek and means ”exaggeration”.

The hyperbole has the ability of never touching neither the first-axis nor the second-axis. We say that it approaches the axis asymptotically, which also is ancient Greek and means ”no coincidence”.

Functions and the four basic arithmetic operations

We may add, subtract, multiply and divide functions by one another.

Let us consider Liz and Peter who in a year earn money in different ways. They work in Denmark, so the currency is in Danish kroner.

Peter has a fixed salary every month:

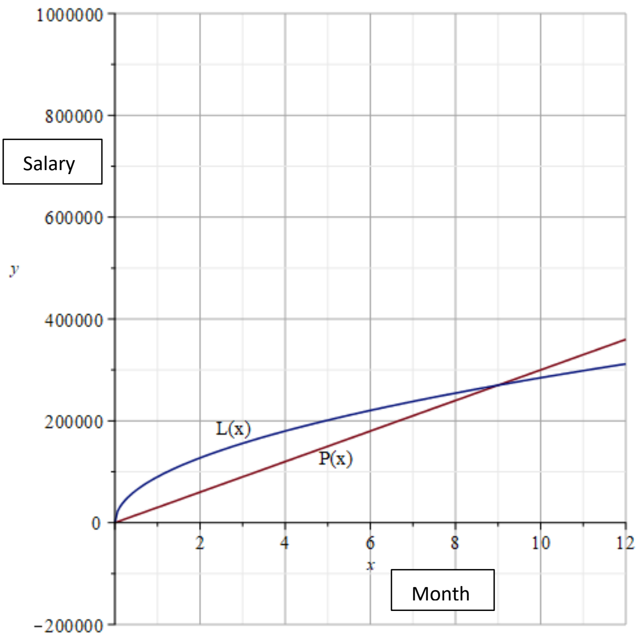
salary = salary per month · number of months =>

$P(x) = 30.000 \cdot x$

Liz earns a lot at the beginning of the year and less later on. It fits approximately this function:

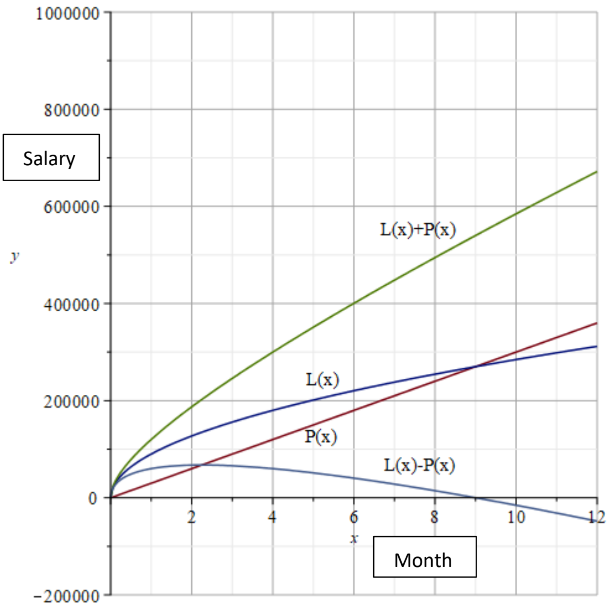
$L(x) = 90.000 \cdot x^{½}$ (she earns 90.000 the first month)

The salary-functions look this way in a diagram:



The following diagram shows how much they earn together (ca. 660.000 kr. for the whole year). The curves have been added.

It also shows how much Liz earns more than Peter. Curve L(x) minus curve P(x). Liz earns most until month nine, the rest of the year she earns less than Peter.

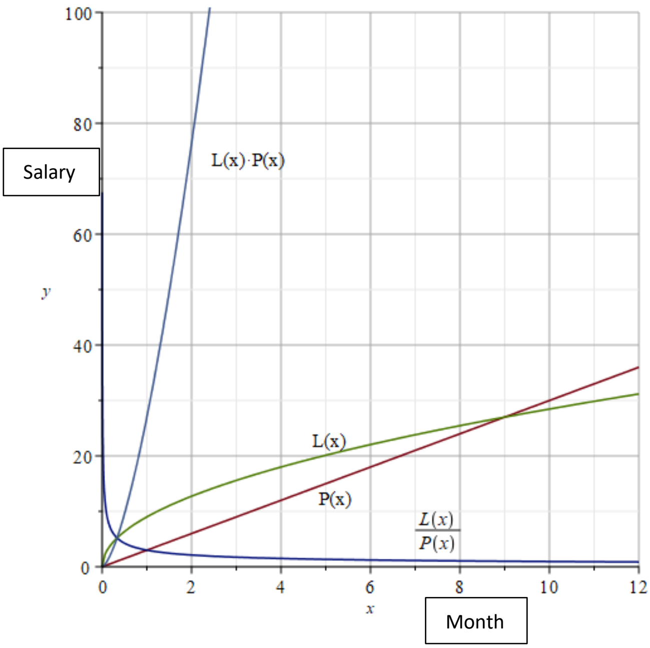


In this case it is not interesting to multiply the two functions. We only get a steep curve that does not give useful information. However, it is shown in the next diagram.

It is more interesting to see how much they earn relative to one another. Here we chose to see how much Liz earns relative to Peter: $\frac{L(x)}{P(x)}$

For a start Liz earns much more than Peter so the fraction gives big values on the second-axis. After a little time, they earn just about the same so the fraction becomes ca.1, and the curve becomes flat.

The values on the second axis are made coarse (1 000 000 has become 100, etc.), so that we can better see the $\frac{L(x)}{P(x)}$ curve.



Composite functions

We use composite functions if the result of one function afterwards applies in another function. For instance

$f(x) \;=\; 2x + 1$ and $g(x) \;=\; x^2$

Here the composite function, where g is performed first and f thereafter, will be called $f(g(x))$. We say f of g of x [some write $(f \circ g)(x)$].

g is called the inner function, and f is called the outer function.

Here:

$f(g(x)) = 2(x^2) + 1$ thus, g inserted into f.

A condition for composite functions is that the range of the inner function lies within the domain of the outer function.

Inverse functions

Usually we see y as a function of x.

For reverse functions, we see x as a function of y. For instance

$y = 2x + 1$ here we have $y = f(x)$ ⇔

$x = \frac{y-1}{2}$ then we have $x = f(y)$

If we use the normal system of coordinates with x on the first-axis and y on the second-axis, we name the new y: x - and the new x: y (we swap). Rather confusing, but that is how it works:

$y = \frac{x-1}{2}$ which is the inverse function in the same system of coordinates.

In order to avoid confusion of names we may write:

Function $f(x) = 2x + 1$ =>

Inverse function $f^{-1}(x) = \frac{x-1}{2}$

The elevated -1 shows that it is an inverse function. This way, it is clear that f and f^{-1} are inverse of one another.

Only monotonous (i.e. increasing or decreasing) functions have an inverse function. Domain and range are swapped as well.

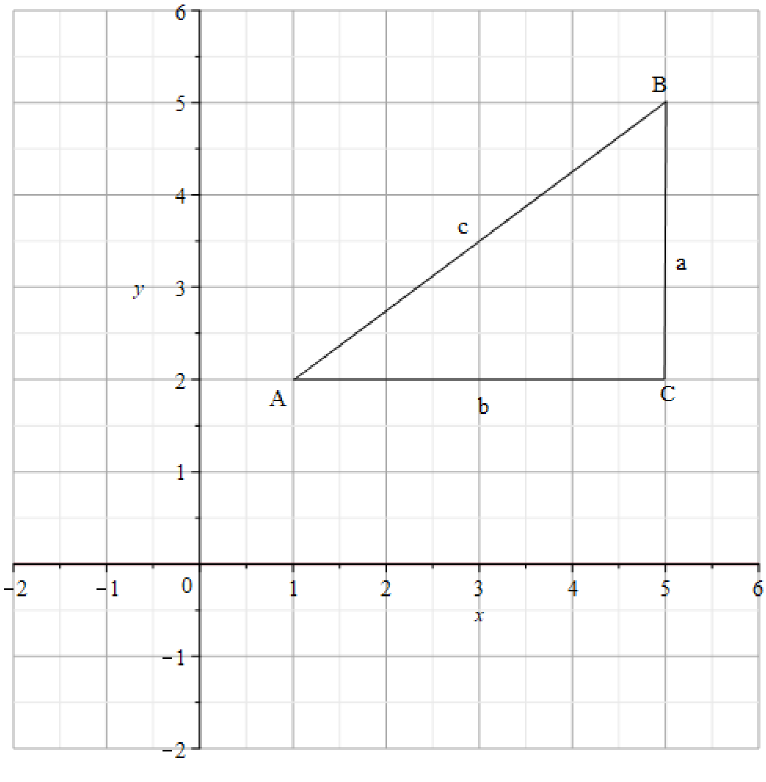
Inverse functions are especially interesting considering sinus, co-sinus and tangent - as well as 10 and log - as well as *e* and ln, which we will see later.

The right-angled triangle

All triangles have an angular sum of 180°. The right-angled triangle has an angle of 90° leaving another 90° for the two others, for instance a triangle with the angles 30°, 60° and 90°.

The right-angled triangle is important within mathematics and it is included in many designs and constructions.

In the diagram is a right-angled triangle called ABC with sides named a, b, c.



The ancient Greek, Pythagoras, found a formula which is valid for right-angled triangles:

$a^2 + b^2 = c^2$

Probably the most used formula of them all.

Seen from angle A, a is the opposite side and b is the adjacent side, and c is the hypotenuse (an old Greek name).

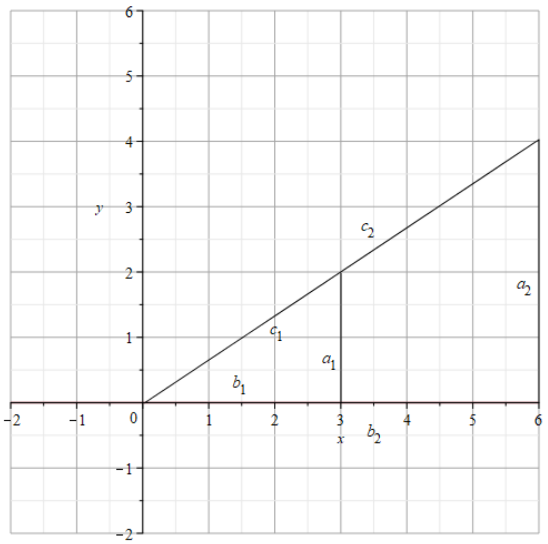
A common right-angled triangle is the 3, 4, 5 triangle, i.e. $a = 3$, $b = 4$ and $c = 5$. Then Pythagoras renders:

$3^2 + 4^2 = 5^2 \quad \Leftrightarrow$

$25 = 25$

which is true. That means we can easily make an angle of 90° by, for instance, having a plank with holes drilled 3 meters apart, another plank with holes drilled 4 meters apart, and a third plank with holes drilled 5 meters apart. Then we place them in a triangle with pegs in the holes, and the biggest angle will automatically be 90°. We can also use three pieces of rope or anything, if only one is 3 long, the other is 4 long, and the third is 5 long, - and it does not have to be meters, it may be feet or any unit.

The next diagram shows a small and a big right-angled triangle. The small triangle has the measures a_1, b_1, c_1 and the big triangle has the measures a_2, b_2, c_2.



If, as shown, the angles are alike, the following applies:

$$\frac{a1}{a2} = \frac{b1}{b2} = \frac{c1}{c2} \qquad \text{and} \qquad \frac{a2}{a1} = \frac{b2}{b1} = \frac{c2}{c1}$$

as can easily be deduced from the diagram.

Put in another way: If a grows, b and c grow correspondingly. If, for instance, a is doubled, b and c are doubled as well. Correspondingly at reduction.

These triangles are one-angled.

The area for all triangles is

$$\text{Area} = \frac{1}{2} \cdot \text{baseline} \cdot \text{height}$$

For a right-angled triangle it is particularly easy, since the baseline is the one cathetus (or leg), and the height is the other cathetus (in Latin the two smaller sides - the legs of the right angle - are called the two Catheti).

For our small triangle, the area is:

$A_1 = \frac{1}{2} \cdot 3 \cdot 2 = 3$

and for the big triangle, the area is:

$A_2 = \frac{1}{2} \cdot 6 \cdot 4 = 12$

Please note that when the side lengths are doubled, the area becomes four times bigger.

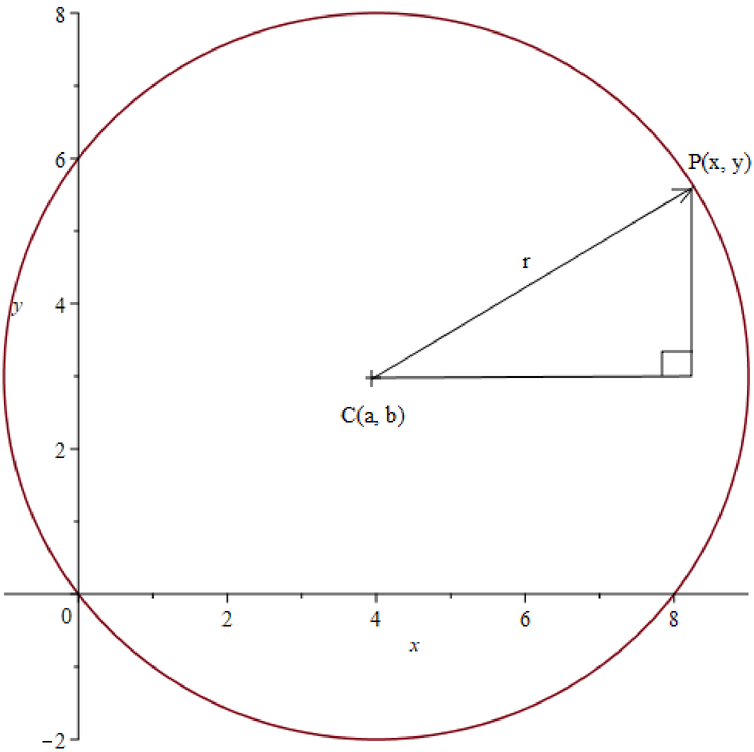
A technical term for all triangles is ”trigonometry” which means triangle measuring.

The circle

The earth is (almost) round. Many celestial bodies are round. The earth rotates around its own axis in a circular motion. Rotating machines do circular motion. Many constructions and designs have circles. The circle is important.

The equation for the circle is surprisingly easy. It is “just” Pythagoras once more.

The diagram shows a circle with centre C(a, b) and radius r hitting the circle in point P(x, y)



The centre C is of course the same regardless of where we are on the circle. P, however, is variable, - the coordinates vary depending on where on the circle we are. The distance (r) to C is the same for all P´s.

We sketch a helping triangle and use Pythagoras:

(horizontal side)^2 + (vertical side)^2 = radius^2 $\Leftrightarrow$

$(x - a)^2 + (y - b)^2 = r^2$

which is the equation of the circle.

Examples

1.

The circle shown has the equation:

$(x - 4)^2 + (y - 3)^2 = 5^2$

Where will our circle intersect the horizontal line $y = 6$?

The line is not shown, but it is seen that the line must intersect in point (0, 6) and (8, 6)

The intersections are where the line equation equals the circle equation, which is solved by two equations with two unknowns:

line $\quad y = 6$

circle $\quad (x - 4)^2 + (y - 3)^2 = 5^2$

line in circle: $\quad (x - 4)^2 + (6 - 3)^2 = 5^2 \quad \Leftrightarrow$

$$x^2 - 8x + 16 + 9 = 25 \quad \Leftrightarrow$$

$$x^2 - 8x = 0$$

here, without c, we may use the zero-solution:

$$x \cdot (x - 8) = 0 \quad \Rightarrow$$

either x or (x - 8) is zero $\Leftrightarrow$

$x = 0$ or $(x - 8) = 0 \Leftrightarrow x = 8$

Combined we find that the line intersects the circle in point
(0, 6) and (8, 6)

which corresponds nicely with our reading.

2.

The ancient Greeks determined the circumference of a circle as:

$O = 2 \cdot \pi \cdot r$

where $\pi \approx 3{,}14$ => $2\pi \approx 6.28$

For our circle it is

$O = 2 \cdot \pi \cdot 5 \approx 31.4$

Also, they determined the area of a circle as

$A = \pi \cdot r^2$

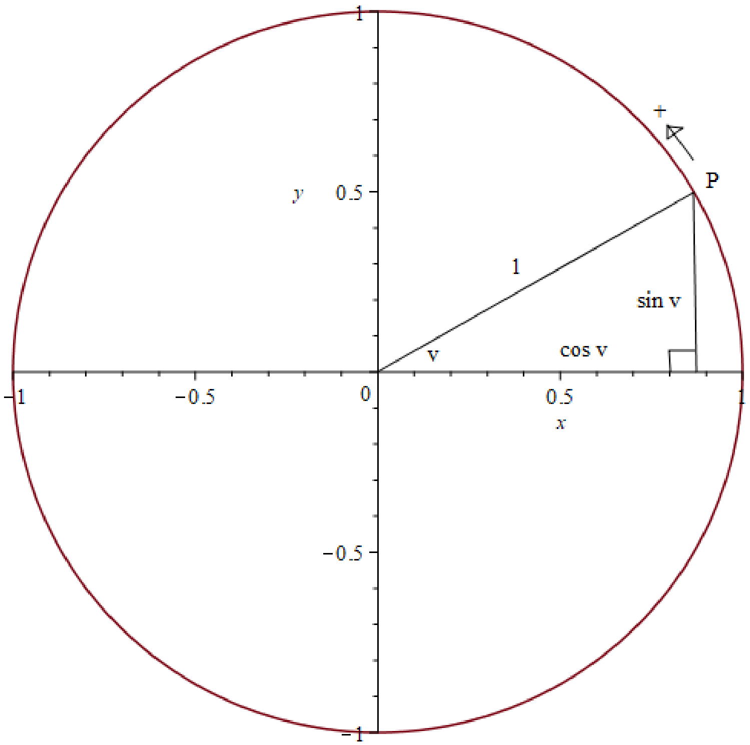
for our circle it is

$A = \pi \cdot 5^2 \approx 78.5$

Sine, cosine and tangent

We imagine that we are playing with a kite in the wind. It does not matter how long the line is (as long as it is constant). We state that the length is 1. It lies horizontally on the ground. Then the wind comes and lifts it in a circular motion. The line length still is 1, so as it rises, it will come closer to us measured along the ground (horizontally). Yet it is still 1 length away from us, only now shared in horizontal and vertical.

Let us see it in a diagram:



We stand at Origo (0, 0). The kite is in point P. The vertical height of the dragon is named *sinus*, and the horizontal distance is named *co-sinus*. Sinus is Latin and stands for ”height of arc”. Co- means “with”, so co-sinus is in context with sinus, since it also depends on sinus. We also clearly see that when the dragon rises, sinus increases and co-sinus decreases. If the kite rises very high, sinus is big and co-sinus is small, - it is almost right above us.

We shorten for sine and cosine, and in calculations we use the abbreviations sin and cos.

sin and cos depend on the angle v (for vertex). (The angle may have all sorts of names). We show it by writing *sin v* for the vertical height of the arc (y-direction), and *cos v* for the horizontal distance (x-direction).

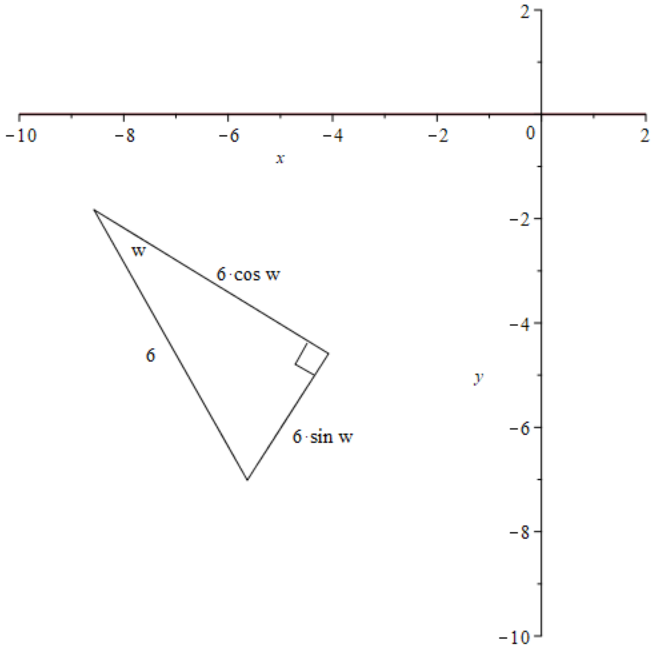
If v becomes bigger P will turn counter clock wise, which is the positive direction of rotation (+ rotation), shown in the diagram by a small arrow and a +.

In other connections clockwise rotation often is plus, but in mathematics +rotation is counter clock wise.

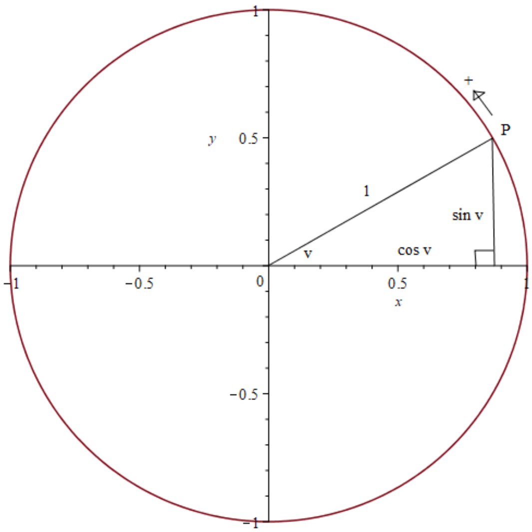
Observed from angle v, the side in front of you (the opposite side) will always be the "sine-side", and the adjacent side that further away ends in a straight (90°) angle will always be the "cosine-side". The long oblique side will always be the hypotenuse.

This is also the case if we move the triangle, change it (yet still straight angled), and turn it.

In the next diagram we show a straight angled triangle, where the hypotenuse (the longest side) is 6 long and the angle is called w. Then the sine-side still is the opposite, and the cosine-side will still be the adjacent side that further away ends in a 90° angle. Only now, they are 6 times longer.



Let us again consider the diagram showing a straight angled triangle in the unit circle (the name of a circle with radius = 1):



The fraction $\dfrac{\sin v}{\cos v}$ informs how big sine is relative to cosine. This fraction is called tangent, tan in brief:

$tan\ v = \frac{\sin v}{\cos v}$

We see that $tan\ v$ gets bigger and bigger (since $sin\ v$ increases and $cos\ v$ decreases) as v goes from 0° towards 90°.

We have already seen a straight angle with a fraction of two catheti, which was:

slope = a = $\frac{\Delta y}{\Delta x}$

and yes, it is the "same". The difference is that the slope is valid for a straight line in a coordinate system, while tangent is valid for a straight angled triangle in any position.

Yet, if we place our straight angled triangle in a coordinate system, we may talk about the slope of our hypotenuse. Thus, for this hypotenuse:

$tan\ v = \frac{\sin v}{\cos v} = \frac{\Delta y}{\Delta x}$ = a = slope

----

If you read an old textbook, it may have another tangent abbreviation: *tg*

And since we talk about notations:

$(\sin v)^2 = \sin^2 v$ $(\cos v)^2 = \cos^2 v$ $(\tan v)^2 = \tan^2 v$

This book has the first notation. Other books and tables may have the other notation.

Examples

1.

Let us start out using Pythagoras on the triangle in the unit circle:

$(\mathit{sin}\ v)^2 + (\mathit{cos}\ v)^2 = 1^2$ called the basic relationship

This equation makes it possible to solve for *sin v*

$(\mathit{sin}\ v)^2 = 1 - (\mathit{cos}\ v)^2 \quad \Leftrightarrow$

$\mathit{sin}\ v = [1 - (\mathit{cos}\ v)^2]^{½}$

where the exponent ½ is used instead of a square root. We will use this relation later on.

2.

Using fine measurement methods on the unit circle, we may find the magnitude of the sine-side and the cosine-side of various angles. These magnitudes lead to functions which are programmed in CAS.

So, for instance we may enter *sin 30°* and get the answer $\frac{1}{2}$

or we may enter *cos 30°* and get the answer $\frac{\sqrt{3}}{2}$

From that we may calculate *tan 30°* $= \frac{1}{\sqrt{3}}$

or we may enter *tan 30°* and get the same answer $\frac{1}{\sqrt{3}}$

Some angles and the corresponding values of sine, cosine and tangent are comfortable to memorize:

Angle	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	-

tangent at 90° would be 1 divided by 0 which cannot be done.

3.

From the table in example 2 we have:

$$sin\ 30° = \frac{1}{2}$$

Also, we may use the table backwards having:

$sin^{-1}\left(\frac{1}{2}\right) = 30°$ (in the US sin^{-1} is called *arcsin* or *invsin*)

or

$$cos\ 45° = \frac{\sqrt{2}}{2}$$

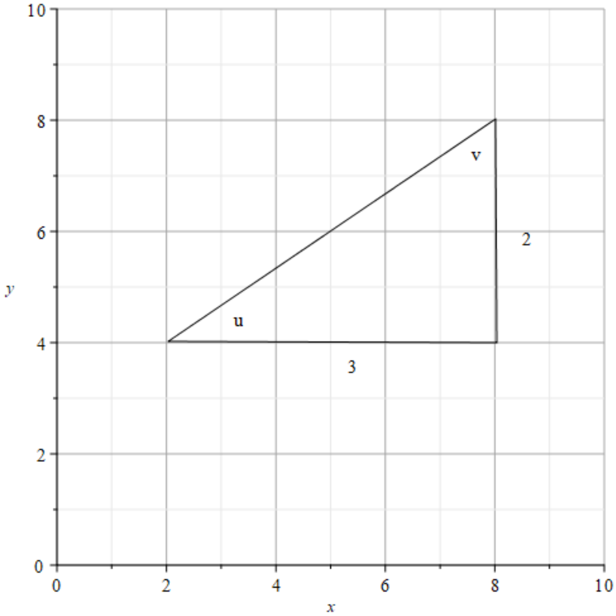
and the inverse function

$cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$ (in the US cos^{-1} = *arccos* = *invcos*)

The inverse functions are programmed into most CAS.

4.

Let us find the angle u and v in this triangle by a calculation commonly used by constructors and designers:



$tan\ u \;=\; \frac{\sin u}{\cos u} \;=\; \frac{2}{3}$

Looking at the table in example 2, we deduce that the angle may be between 30° and 60° (nearest 30). CAS also has the inverse function, so asking for tan^{-1} to $\frac{2}{3}$ renders 33,69°

We find angle v the same way, only now the sine-side is 3 and the cosine-side is 2:

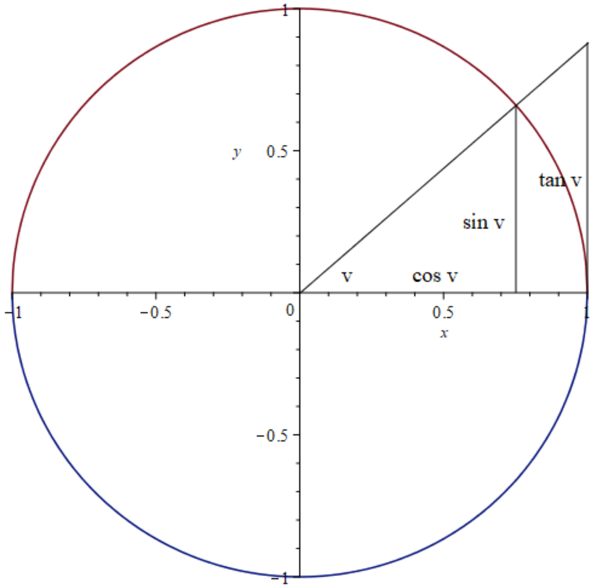
$tan\ v \;=\; \frac{\sin v}{\cos v} \;=\; \frac{3}{2} \qquad \Leftrightarrow$

$\mathrm{v} \;=\; \tan^{-1}\left(\frac{3}{2}\right) \;=\; \arctan\left(\frac{3}{2}\right) \;=\; \mathrm{invtan}\left(\frac{3}{2}\right) \;=\; 56{,}31°$

Controlling: 90° + 33,69° + 56,31° = 180° Ok.

5.

An example, to show from where tangent has its name:



The small triangle and the big triangle are one-angled =>

$$\frac{\sin v}{\cos v} = \frac{\tan v}{1}$$

and using the unit circle we see tan v depicted.

In times before CAS, it was possible to determine tangent this way, yet this is not the definition of tangent, which is:

$$\tan v = \frac{\sin v}{\cos v}$$

Radian

Radius radiates from the centre to the periphery, thus the name.

Radian is nature's own way of measuring an angle. Man decided to measure an angle in degrees, but also, it is measured in radians:

The ancient Greeks found that the circumference of a circle is proportional with the radius of the circle. They called the proportionality factor: 2π

Circumference $= 2 \cdot \pi \cdot$ radius $\Leftrightarrow$

$O = 2\pi \cdot r$

They also found $\pi \approx 3{,}14 \;\; \Rightarrow \;\; 2\pi \approx 6{,}28$

Thus, radius is the variable determining the size of a circle's circumference. If r doubles, O will double too, etc.

So, one tour round a circle is 2π radii regardless of the size of the circle. This, we now call 2π radian, or briefly: 2π rad

One tour round a circle is also a tour of 360° regardless of the size of the circle. Therefore:

2π radian $=$ 360 degrees

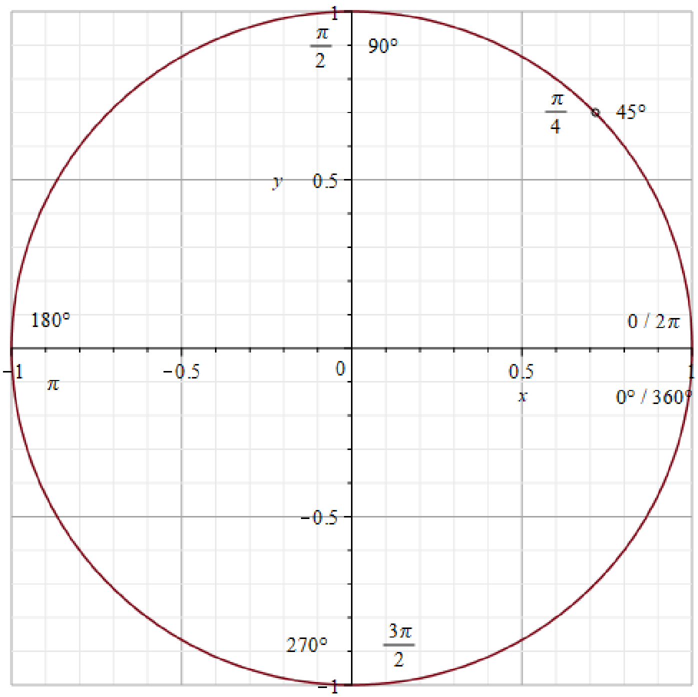
which may be down parted to for instance:

π rad $= 180°$ $\Leftrightarrow$

$\frac{\pi}{2}$ rad $= 90°$ $\Leftrightarrow$

$\frac{\pi}{4}$ rad $= 45°$

and so on. See also the figure:



Proportional calculation leads to this conversion formula:

$$\frac{angle\ in\ radian}{2\pi} = \frac{angle\ in\ degrees}{360}$$

Examples

1.

An angle v is 30°, what is it in radians?

Answer:

$$\frac{angle\ in\ radian}{2\pi} = \frac{angle\ in\ degrees}{360} \quad \Leftrightarrow$$

$$v_{rad} = 2\pi \cdot \frac{30}{360} \quad \Leftrightarrow$$

$$v_{rad} = \frac{\pi}{6}\ \text{rad}$$

Usually, we do not render a decimal number. We leave π in order to be precise and to show, that we are using radians, - even if we also state, rad

2.

An angle w is 60°, what is it in radians?

Answer:

$$\frac{angle\ in\ radian}{2\pi} = \frac{angle\ in\ degrees}{360} \qquad \Leftrightarrow$$

$$w_{rad} = 2\pi \cdot \frac{60}{360} \qquad \Leftrightarrow$$

$$w_{rad} = \frac{\pi}{3}\,\text{rad}$$

3.

An angle is 1 rad, what is it in degrees?

Answer:

$$\frac{angle\ in\ radian}{2\pi} = \frac{angle\ in\ degrees}{360} \qquad \Leftrightarrow$$

$$u_{deg} = \frac{360 \cdot 1}{2\pi} \qquad \Leftrightarrow$$

$$u_{deg} \approx 57{,}3°$$

Here we usually write a decimal number.

4.

An angle φ (fi) is π rad, what is it in degrees?

Answer:

$$\frac{angle\ in\ radian}{2\pi} = \frac{angle\ in\ degrees}{360} \quad \Leftrightarrow$$

$$\varphi_{deg} = \frac{360 \cdot \pi}{2\pi} \quad \Leftrightarrow$$

$$\varphi_{deg} = 180°$$

Angle and circular arc length

Let us again look at the formula for the circumference of a circle:

$O = 2\pi \cdot r$ O is the special arc length called the circumference

and divide by 2 on either side

$\frac{O}{2} = \pi \cdot r$ which is half a round

then we divide by, for instance, 3 on either side

$\frac{O}{6} = \frac{\pi}{3} \cdot r$ and the arc length becomes less

These three expressions gathered in a common equation:

arc length = angle (in rad) · radius

angle and arc length are directly proportional.

Example

5.

What is the arc length of 45° of a circle with radius 9 meters?

We alter from 45° to radians:

$$\frac{angle\ in\ radian}{2\pi} = \frac{angle\ in\ degrees}{360} \quad =>$$

angle in rad $= 2\pi \cdot \frac{45}{360}$ $\Leftrightarrow$

angle in rad $= \frac{\pi}{4}$

and: arc length $= \frac{\pi}{4} \cdot 9 \approx 7{,}07$ meters

End of chapter

Let us end this chapter by showing a table from the former chapter (sine, cosine and tangent), - only now expanded with angles in radians:

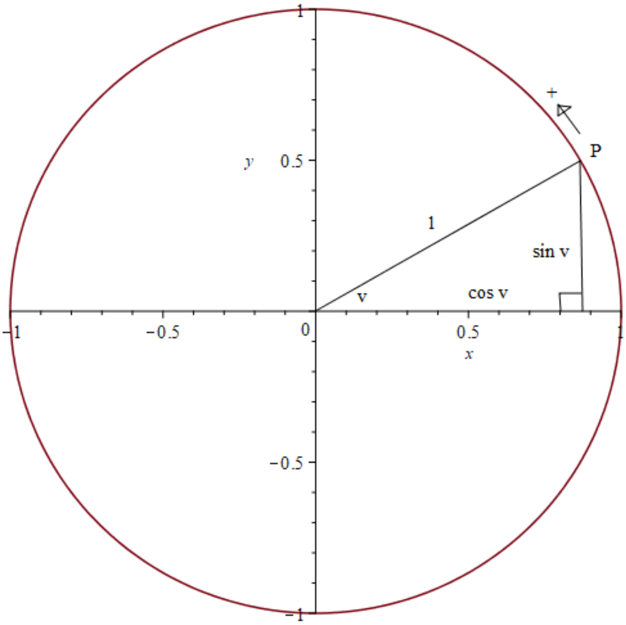
Angle	0°	30°	45°	60°	90°
Angle	*0 rad*	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	-

tangent at 90° would be 1 divided by 0, which cannot be done.

In some books and in some tables, v is an angle in degrees, and x is an angle in radians. Often we need to be aware of different names for things alike.

The sine function and the sine oscillation

We will look at the unit circle again:



We call it the unit circle because radius is 1.

If angle v is 0, point P is on the x-axis with the coordinates (1, 0).
Then $\cos v = 1$ and $\sin v = 0$

if v increases, $\sin v$ will increase, and $\cos v$ will decrease.

if v = 90° ($\frac{\pi}{2}$): $\cos v = 0$ and $\sin v = 1$

if we continue rotation in the + direction (counter-clockwise) in 2.quadrant, $\cos v$ becomes more negative and $\sin v$ less positive

at v = 180° (π): $\cos v = -1$ and $\sin v = 0$

in 3.quadrant $\cos v$ becomes less negative, and $\sin v$ becomes more negative

at v = 270° ($\frac{3\pi}{2}$): $\cos v = 0$ and $\sin v = -1$

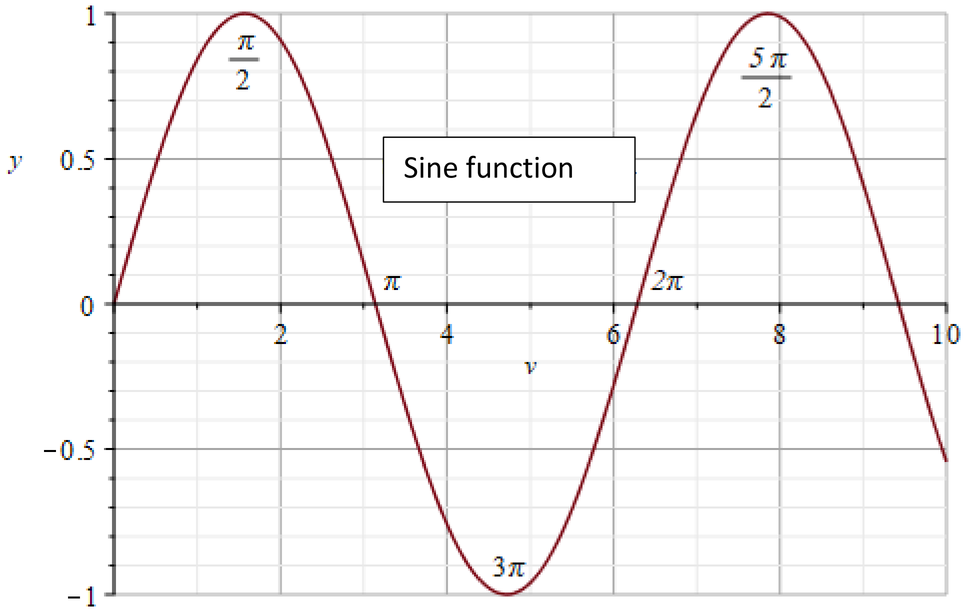
in 4.quadrant *cos v* becomes more positive, and *sin v* becomes more negative

at v = 360° (2π) we start over.

The way *sin v* varies, can be written in a function:

$y = \sin v$

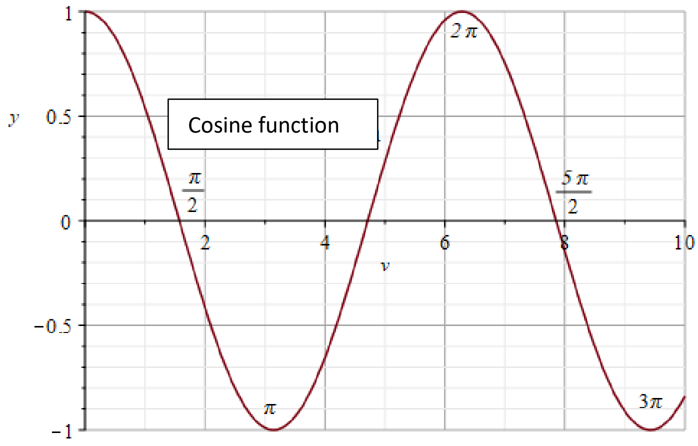
and we can display the function in a coordinate system with *v* in radians on the first-axis and *sin v* on the second-axis:



The curve fluctuates about the neutral axis (here the first-axis). The maximum fluctuation is called the amplitude (here +1 and -1). One rotation from 0 to 2π is called a cycle or a period.

We also display the cosine function

$y = \cos v$



Which is alike only moved $-\frac{\pi}{2}$ along the first-axis.

So, we only consider the sine function.

As we have seen, the sine function may be used to find angles and arc lengths of circle segments, and together with cosine and tangent we can do calculations on straight angled triangles, which is important within geometry. We can state, that the sine function combines circles, angles, arc lengths and straight angled triangles.

Yet, the sine function holds more:

Sine oscillation

Many things rotating or fluctuating follow the sine function. Repeating events. We name them sine oscillations. Sine oscillations appear in nature as well as in technics, and especially the technical oscillations demand an expansion of the equation.

We find sine oscillations in nature in for instance our pulse and breath, temperature fluctuations over the year, the 24 hour rhythm, tide, sound, light, etc.

Examples for technical oscillations are rotating machinery, sound technique, music instruments, light technique etc.

Sine oscillations apply when something fluctuates up and down, forth and back, rotates, etc.

Let us consider something rotating with a constant speed, - for instance the earth´s rotation around its own axis. Then arc length and time are related:

In 24 hours the earth turns the angle 2π radians and the arc length $2\pi r$. In 1 hour the earth turns $\frac{2\pi}{24}$ radians, and the arc length $\frac{2\pi r}{24}$ etc.

So, angle and arc length are proportional.

We now need a physical size called the angular velocity (ω), which defines as the angular turn in radians divided by time in seconds (t):

angular velocity $= \frac{angle}{time}$

and in symbols

$\omega = \frac{v}{t}$ ⇔ v is the angle

$v = \omega t$ ω is the Greek omega

Now our angle is ωt and the sine function is thus a function of time, t, and looks this way:

$f(t) = \sin \omega t$

In techniques we may need to change the angle. It is written as:

$f(t) \;=\; \sin\,(\omega t + \varphi)$ φ is the Greek letter fi.

Now the angle is $(\omega t + \varphi)$

We may also need to change the amplitude. We do so, by multiplying by A, the size of the amplitude. A is measured from the neutral line to the maximum top or bottom.

Finally, we may need to move the sine curve up (or down) in the diagram. Therefore, we add k. If k is positive, the curve is moved upwards, and if k is negative, it is moved downwards:

$f(t) \;=\; \mathrm{A} \cdot \sin\,(\omega t + \varphi) + k$

This is the expanded sine function, or the *equation for a sine oscillation.*

These sine oscillations are also named harmonic oscillations.

Examples

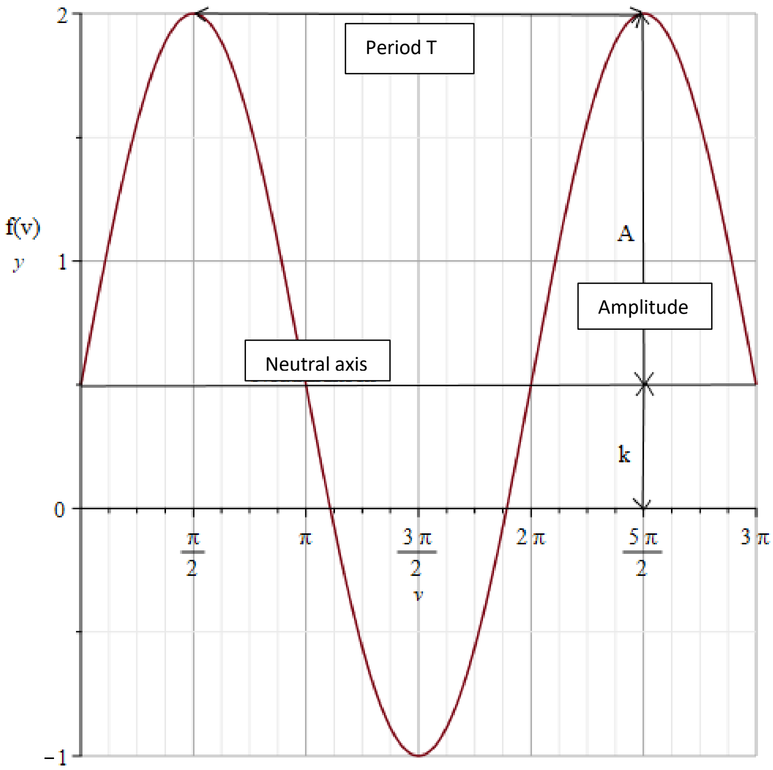
1.

This diagram shows a partly expanded sine function, where the curve continues to depend of the angle v, and where:

A = 0,5 k = 0,5 =>

$f(v) \;=\; 1{,}5 \cdot \sin(v) + 0{,}5$

v on the first axis and f(v) on the second axis.



It reads, that the neutral axis now is for f(v) = 0,5 since k = 0,5 and the amplitude A equals 1,5

2.

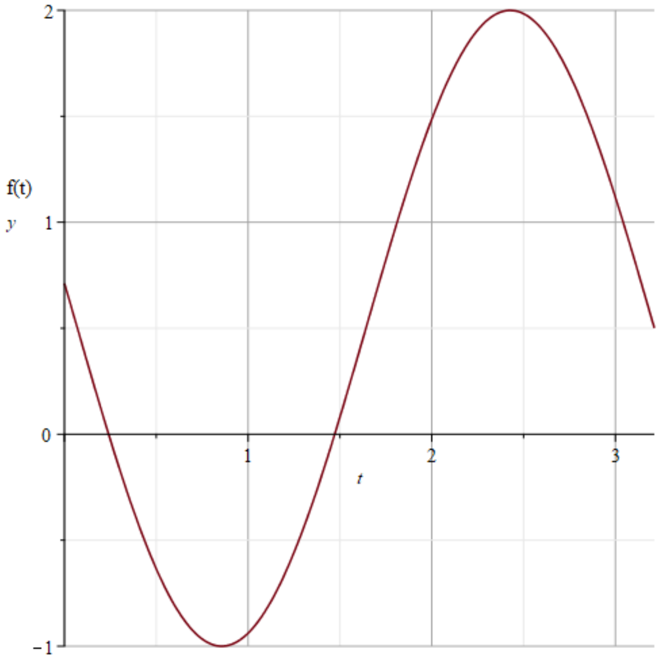
Now we wish to consider the sine function related to time instead of angle. We do so by insertion of numerical values for ω and φ.

The diagram displays a fully expanded sine function, where:

A = 1,5   ω = 2   φ = 3   k = 0,5   =>

$f(t) \;=\; 1.5 \cdot \sin(2t + 3) + 0.5$

with t on the first axis and f(t) on the second axis.



We note that the values on the second axis show an unchanged oscillation from -1 to 2, while the values on the first axis have changed from angle to time. This is due to the values inserted for ω and φ.

ω and φ put together in the fraction $-\frac{\varphi}{\omega}$ move the curve in the x-direction without changing its shape. $-\frac{\varphi}{\omega}$ is called the phase shift. This is proven below:

Proofs

The angle $(\omega t + \varphi)$ for one oscillation lies between 0 and 2π radians:

$0 \leq (\omega t + \varphi) \leq 2\pi \quad \Leftrightarrow$

$-\varphi \leq \omega t \leq 2\pi - \varphi \quad \Leftrightarrow$

$-\frac{\varphi}{\omega} \leq t \leq \frac{2\pi-\varphi}{\omega}$

So the initial value has changed from 0 to $-\frac{\varphi}{\omega}$ which therefore is called the phase shift.

The calculation also shows that one oscillation goes from $-\frac{\varphi}{\omega}$ to $\frac{2\pi-\varphi}{\omega}$. We see that by having:

end minus start $\Leftrightarrow$

$\frac{2\pi-\varphi}{\omega} - (-\frac{\varphi}{\omega}) = \frac{2\pi}{\omega}$ which is named the period, T

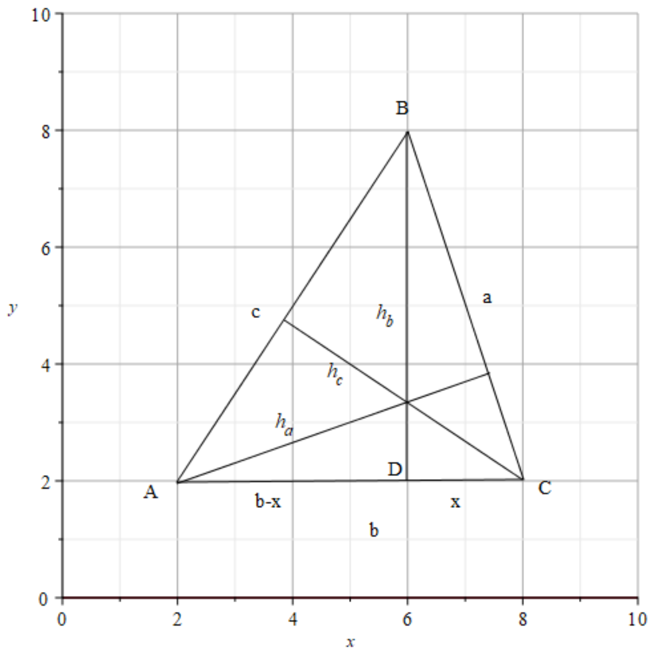
$T = \frac{2\pi}{\omega}$

Often t is time so T is for one whole oscillation = one cycle = one period.

The not right angled triangles

Also called the arbitrary triangles.

This chapter also comprises the isosceles and equilateral triangles, which usually are not considered arbitrary.



For all triangles the angular sum is 180°

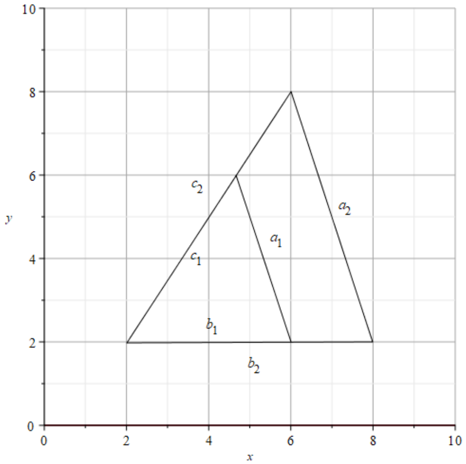
The area of all triangles is

$$\text{Area} = \frac{1}{2} \cdot \text{base line} \cdot \text{height}$$

A height in a triangle leads from one corner perpendicularly to the opposite side.

For all one-angled triangles, it applies that

$$\frac{a1}{a2} = \frac{b1}{b2} = \frac{c1}{c2} \qquad \text{and} \qquad \frac{a2}{a1} = \frac{b2}{b1} = \frac{c2}{c1}$$



Yet, Pythagoras does *not* apply for arbitrary triangles. Instead, the sine rules and the cosine rules apply:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad\qquad c^2 = a^2 + b^2 - 2a \cdot b \cdot \sin C$$

Proofs

The sine rules:

The area of a triangle may be written in three ways:

$$A = \frac{1}{2} \cdot a \cdot h_a = \frac{1}{2} \cdot b \cdot h_b = \frac{1}{2} \cdot c \cdot h_c$$

Instead, the heights may be seen as sine sides in the smaller inner triangles (see the former figure). For instance

$$h_a = \sin B \cdot c \qquad \Rightarrow$$

$$\frac{1}{2} \cdot a \cdot \sin B \cdot c = \frac{1}{2} \cdot b \cdot \sin C \cdot a = \frac{1}{2} \cdot c \cdot \sin A \cdot b \qquad \Leftrightarrow$$

$$a \cdot \sin B \cdot c = b \cdot \sin C \cdot a = c \cdot \sin A \cdot b \qquad \Leftrightarrow$$

and when we divide all parts of the equation by $a \cdot b \cdot c$.

$$\frac{\sin B}{b} = \frac{\sin C}{c} = \frac{\sin A}{a} \qquad \Leftrightarrow$$

or reciprocal as stated in some tables

$$\frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a}{\sin A}$$

The sine rules are the easiest, but if they prove to be insufficient, the cosine rules will probably apply

The cosine rules

The line segment b in the first diagram may be split in b-x and x and since the inner, smaller, triangles are straight, we can use Pythagoras

first triangle BCD

$x^2 + h_b^2 = a^2 \qquad \Leftrightarrow \qquad h_b^2 = a^2 - x^2$

then triangle ABD

$(b - x)^2 + h_b^2 = c^2 \qquad =>$

$b^2 + x^2 - 2bx + (a^2 - x^2) = c^2 \qquad h_b^2$ inserted

x is the cosine side for angle C in triangle BCD

$x = a \cdot \cos C \qquad =>$

$a^2 + b^2 - 2a \cdot b \cdot \cos C = c^2 \qquad$ x inserted

We could also have considered the line segments a and b which would yield:

$b^2 + c^2 - 2b \cdot c \cdot \cos A = a^2$

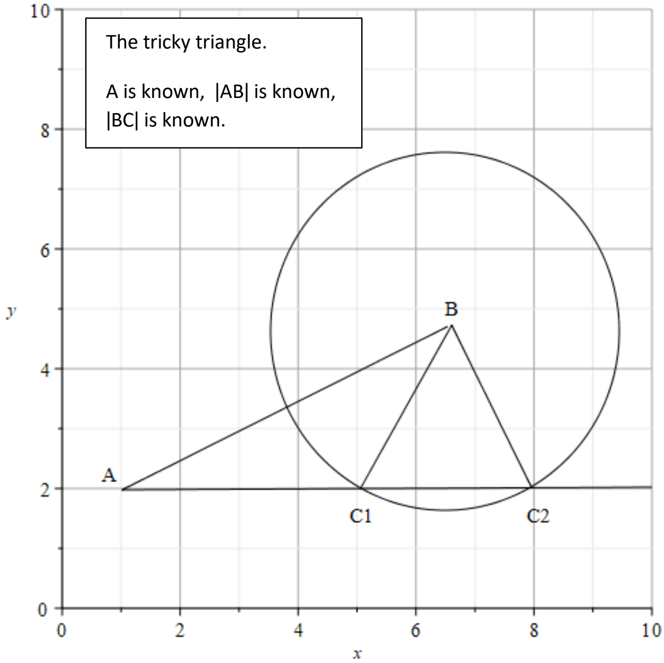
$a^2 + c^2 - 2a \cdot c \cdot \cos B = b^2$

So, there are three cosine rules, but usually tables only show one of them.

More theory

We need to know three things about a triangle to be able to construct it. There are, however, two exceptions:

- If we know all three angles, the triangle may have all sizes.
- If we know two side lengths and an angle which is not between the two, there might be a problem, which is illustrated in the figure:



We sketch angle A. Mark point B. |BC| is now the radius for a compass, and we see that C may have two positions.

This error is unfortunately rather common, for instance in designing a rafter. The designer meant C1 but had C2.

Examples

For a triangle B = 82°, b = 14 and c = 13,1

What is C?

We use the part of the sine rules where we have information:

$$\frac{c}{\sin C} = \frac{b}{\sin B} \qquad \Leftrightarrow$$

$$\sin C = \frac{c \cdot \sin B}{b} \qquad =>$$

$$\sin C = \frac{13{,}1 \cdot \sin 82°}{14} \qquad \Leftrightarrow$$

$$C = \sin^{-1}\left(\frac{13{,}1 \cdot \sin 82°}{14}\right) \qquad \Leftrightarrow$$

$$C = 67{,}9°$$

Side a?

A is 180° - 82° - 67,9° = 30,1°

We can use the sine rules one more time, but we chose the cosine rule

$$b^2 + c^2 - 2b \cdot c \cdot \cos A = a^2 \qquad =>$$

$$a^2 = 14^2 + 13{,}1^2 - 2 \cdot 14 \cdot 13{,}1 \cdot \cos 30{,}1° \qquad \Leftrightarrow$$

$$a^2 = 50{,}27 \qquad \Leftrightarrow$$

$$a = 7{,}09$$

Exponential functions

Exponential functions have a positive constant (here called a) as base number and x as exponent

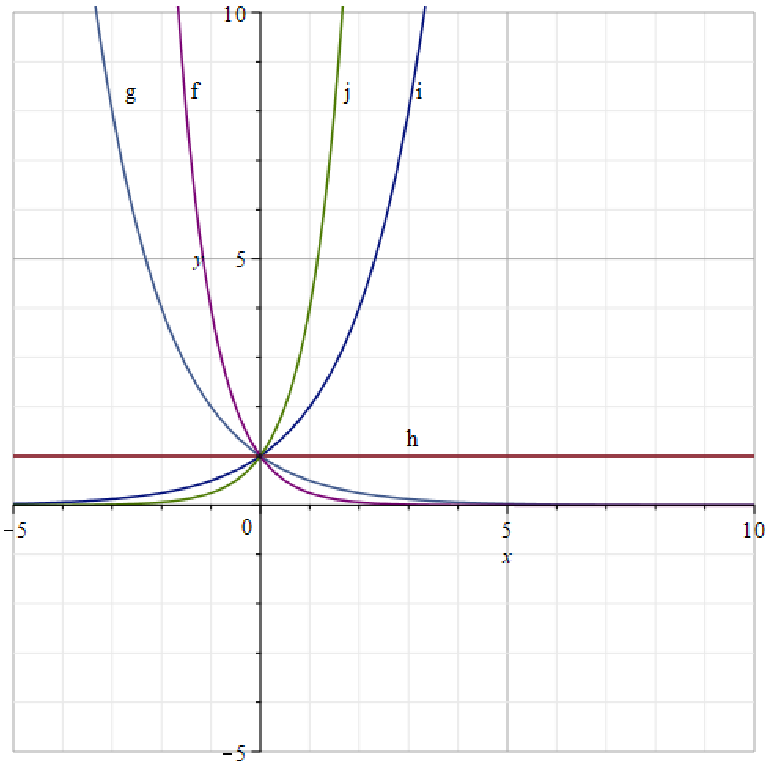
$y = a^x$ or

$f(x) = a^x$

In the diagram:

$f(x) = \left(\frac{1}{4}\right)^x$ $g(x) = \left(\frac{1}{2}\right)^x$ $h(x) = 1^x$

$i(x) = 2^x$ $j(x) = 4^x$



This time x is omitted in f(x) and shortened to f, g, etc. It is allowed since it is clear that x is on the first axis.

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Exponential functions have domain in the interval]-∞;∞[and range in the interval]0;∞[and are therefore placed in the first and second quadrant only. The first quadrant is usually the interesting one.

Generally, an exponential function is decreasing (seen in the +x direction) when a is between 0 and 1, - and increasing when a is bigger than 1.

The h function has $a = 1$, so it is horizontal and separates decreasing from increasing.

Also, note that all exponential functions go through point (0, 1), so in the first quadrant they will all start here.

Exponential functions are interesting, because they often apply in nature, particularly the one with the base number 2.7183… or approximately 2.72. This base number is called e after a known mathematician by the name Euler:

Euler's number $= e \approx 2.72$

Exponential functions with base number e are shown in the next diagram.

We find $e \approx 2.72$ by reading f(x) at $x = 1$. The point is marked $(1, e)$

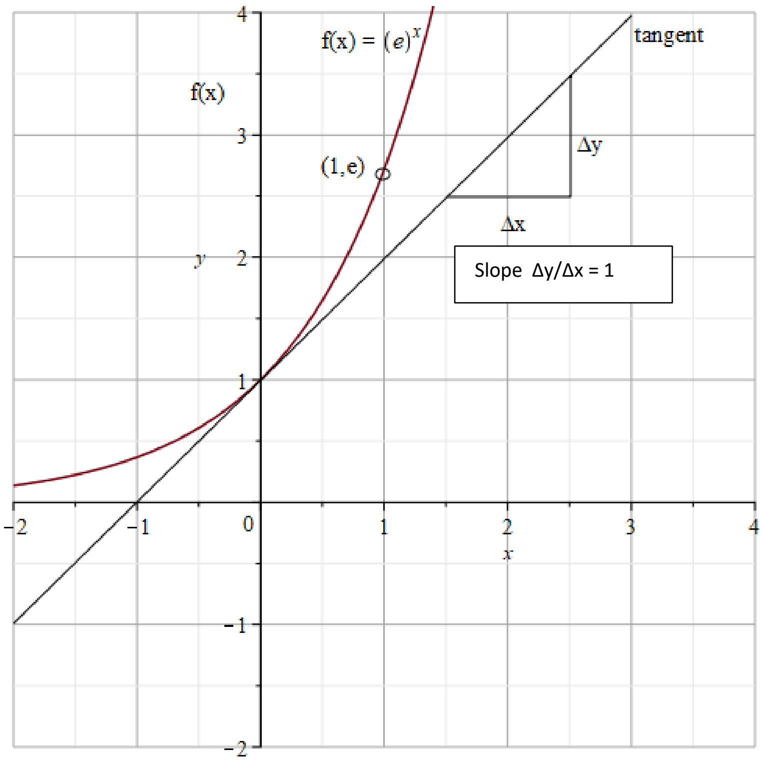
Why is base number e so interesting? It is because of the slope of the curve, which we need to explain first:

The slope informs how much a function increases (the slope is positive) or decreases (the slope is negative).

The slope changes, so if we will find it in a certain point, we put a tangent to the point and find its slope, which also is the curve's slope right there. At the starting point (0, 1) where f(x) is 1, our exponential function e^x has the slope 1.

Now, we can give an answer:

The exponential function with base number *e* is so interesting because for f(x) = 1 the slope also is 1. For f(x) = 2 the slope also is 2. For f(x) = 3 the slope is 3, etc.



So for all exponential *e* functions we have:

f(x) $= e^x$ and tangent slope $= e^x$

No other function has this ability that the function value and the tangent slope is the same, namely e^x. An ability found in many connections for biological growth of bacteria, plants, animals,……, and physical as well as chemical processes.

Therefore, the function is named: ”the natural exponential function”, and the base number *e* (Euler´s number) is one of the most significant constants in mathematics.

Much more about this in Part 3.

More theory 1

First we consider the ordinary exponential function with the variable a as base number (the one we started out with). Just like when we earlier wanted to expand the sine function, we now want to expand the exponential function so it may be widely used. We expand from

$f(x) = a^x$ the exponential function

to

$f(x) = b \cdot a^{kx}$ the function for exponential growth

where b gives the curve another point of intersecting the second axis (or starting point, if we only consider the first quadrant), and k makes the curve more or less steep. Alt. $f(x) = b \cdot c^x$ for $a^k = c$

Example 1

Let us consider an example from economy:

$f(x) = b \cdot a^{kx}$ is rearranged

$K_n = K_0 \cdot (1 + r)^n$ which is the interest formula

Here f(x) is now called K_n which is a future value b is called K_0 which is the initial value

a is called (1 + r) where r is the interest rate

kx is now called n, the number of terms, which is the time period of the actual interest rate (often in number of years).

Using this equation, we may for instance calculate the future value of money we lend in the bank today and pay back 5 years from now. If we lend 10.000 pounds with an interest rate of 4% per year in 5 years, we then will owe:

$K_5 = 10.000 \cdot (1 + 0{,}04)^5 = 12.167$ pounds

More theory 2

In the same manner we now consider the natural exponential function with *e* as base number and expand from

$f(x) = e^x$ the natural exponential function

to the

$f(x) = b \cdot e^{kx}$ *e* function for exponential growth

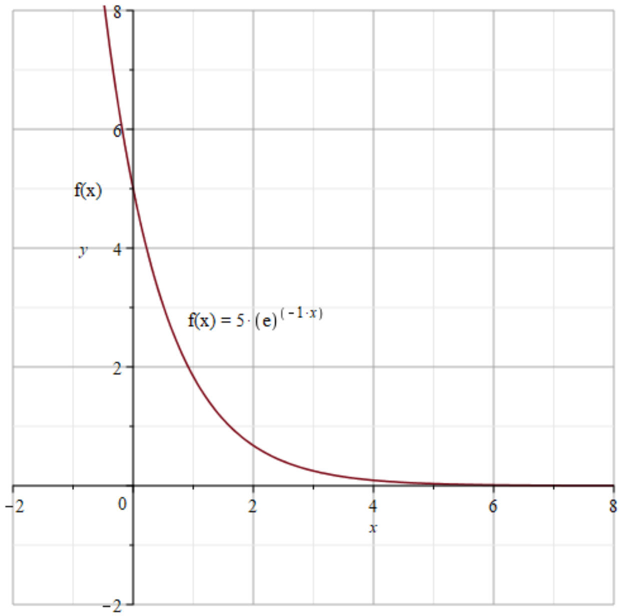
where b gives the curve another point of intersecting the second axis (or starting point, if we only consider the first quadrant), and k makes the curve more or less steep. k is negative at negative growth.

Example 2

The diagram shows an *e* function for exponential growth, where $b = 5$ and $k = -1$.

the negative k makes the curve exponentially decreasing.

An example might be decay of a radioactive matter, here with time on the first axis and radioactivity on the second axis. The curve would start in point (0, 5), unless we want to go back in time.



Logarithm functions (log and ln)

Logarithm is Latin and means something like ”arithmetic numbers”.

Here we calculate in a new way.

The four basic arithmetic operations can be used for most, but we need more. The first time, we saw something new, were the functions sin, cos and tan (going from angle to distance), and their inverse functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ - or arcsin, arccos, arctan - or invsin, invcos, invtan (going from distance to angle).

Now we again have to calculate in a new way.

We need a tool to find x in exponential equations like

$y = 10^x$ and $y = e^x$

The 10s logarithm

Let us consider this row of numbers with powers of 10. Below we just write the exponents:

10^x	$\uparrow$	…	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3	10^4	…	$\downarrow$	$\log 10^x$
x		…	-2	-1	0	1	2	3	4	…		x

It is clear that we go from row 2 to row 1 by saying:

x is transferred to 10^x or

$x \longrightarrow 10^x$

When we go from row 1 to row 2, we now decide to say:

$\log 10^x \longrightarrow x$ so, we define that $\log 10^x = x$

In words: The logarithm of 10^x is x, - and then we may insert a number instead of x.

Now we can solve the equation

$y = 10^x$ by having the logarithm on either side

$\log y = \log 10^x \quad \Leftrightarrow$

$\log y = x \quad \Leftrightarrow$

$x = \log y$

If y is a known number, for instance 100 (= 10^2), we can read in the table we just made that $\log 10^2 = 2$ and finish by having

$x = 2$

Our table only contains a few numbers, but fine tables are programmed into CAS. For instance, we may enter:

$\log 37 \approx 1.57$

Please note that all numbers in row 1 are positive. We can only find the logarithm of numbers bigger than zero (> 0).

There are more advantages, since it gives us the opportunity to shift between row 1 and 2. Multiplication and division in row 1, become addition and subtraction in row 2, - and vice versa.

Example I. $10^2 \cdot 10^3$ in row 1 become 2 + 3 in row 2. 2 + 3 is 5. Returning to row 1 we find the answer 10^5

Example II. $10^2 : 10^3$ in row 1 becomes 2 - 3 in row 2. 2 - 3 is -1. Returning to row 1, we find the answer 10^{-1}

We rarely need the other way round, but let us try:

-1 + 4 in row 2 become 10^{-1} and 10^{4} in row 1. $10^{-1} \cdot 10^{4}$ is 10^{3}. Returning to row 2, we read the answer: 3

It works.

In times before CAS, we used a slide rule which is based on the 10s logarithm. Today we have CAS, so now logarithms are mainly used to solve exponential equations just as we started out with.

The calculation rules for 10s logarithms are:

Instead of numbers in example I

$\log(10^2 \cdot 10^3) \;=\; 2 + 3 \;=\; \log 10^2 + \log 10^3$

we can use letters:

$\log(a \cdot b) \;=\; \log a + \log b$ which is rule number 1

Instead of numbers in example II

$\log(10^2 : 10^3) \;=\; 2 - 3 \;=\; \log 10^2 - \log 10^3$

we can use letters

$\log\left(\frac{a}{b}\right) \;=\; \log a - \log b$ which is rule number 2

The last rule is, if the power number not is 10 (or *e*). We derive it

$\log a^x \;=\; \log(10^{\log a})^x \;=\; \log 10^{x \log a} \;=\; x \cdot \log a \qquad \Leftrightarrow$

$\log a^x \;=\; x \cdot \log a$ which is rule number 3

There are only these calculation rules for logarithms.

The natural logarithm

Or the *e*-logarithm.

Everything is just like the 10s logarithm, only now the base number is *e*

e^x	$\uparrow$	… e^{-2}	e^{-1}	e^0	e^1	e^2	e^3	e^4 …	$\downarrow$	$\ln e^x$
x		… -2	-1	0	1	2	3	4 …		x

and we call it ln x which stands for ”logarithm natural” or more idiomatic: ”the natural logarithm” since it is found in nature just like its base number *e*. It is clear that we go from row 2 to row 1 by saying:

x is transferred to e^x or

$x \longrightarrow e^x$

When we go from row 1 to row 2, we now decide to say:

$\ln e^x \longrightarrow x$ so, we define that $\ln e^x = x$

In words: The natural logarithm of e^x is x, - and then we may insert a number instead of x.

Now we can solve the equation

$y = e^x$ by having ln on either side

$\ln y = \ln e^x \quad \Leftrightarrow$

$\ln y = x \quad \Leftrightarrow$

$x = \ln y$

If y is a known number, for instance e^2, we can read in the table, we just made, that $\ln e^2 = 2$ and finish by having

$x = 2$

Our table only contains a few numbers, but fine tables are programmed into CAS. For instance, we may enter:

$\ln 37 \approx 3.61$

Please note that all numbers in row 1 are positive. We can only find the logarithm of numbers bigger than zero (> 0).

The calculation rules are

$\ln(a \cdot b) = \ln a + \ln b$ which is rule number 1

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ which is rule number 2

$\ln a^x = x \cdot \ln a$ which is rule number 3

More theory

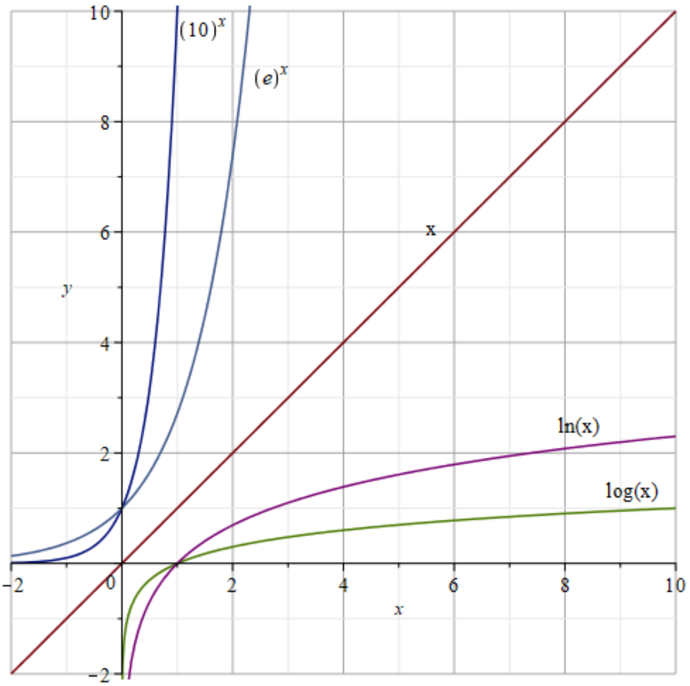
It appears above, that 10^x and $\log x$ are inverse functions

$\log 10^x = x$ and $10^{\log x} = x$

Correspondingly, e^x and $\ln x$ are inverse functions

$\ln e^x = x$ and $e^{\ln x} = x$

We can display it in a diagram, where $y = x$ is also shown for comparison, and we see that $y = x$ is the line of symmetry, so, the curves may be mirrored in this symmetry line.



Here we only write the right side of the expression. It cannot be misunderstood so it is allowed.

The exponential functions/curves lean more and more, while the logarithm functions/curves lean less and less when x increases.

Other notations for the logarithm function to the power of 2:

$(\log x)^2 \ = \ \log^2 x \qquad\qquad (\ln x)^2 \ = \ \ln^2 x$

In this book, we use the first notation. Some other books and tables may use the other notation.

Examples using calculation rule 1

The sign $\approx$ means CAS was used.

$\log 3 + \log 4 \ = \ \log (3 \cdot 4) \ = \ \log 12 \ \approx \ 1.079$ (log 12 is precise)

$\log 4 + \log 25 = \log (4 \cdot 25) = \log 100 = 10$

$\log \left(\frac{2}{3}\right) + \log \left(\frac{3}{4}\right) = \log \left(\frac{2}{3} \cdot \frac{3}{4}\right) = \log \left(\frac{2}{4}\right) \approx -0.301$

$\log (10 \cdot \sqrt{10}) = \log 10 + \log \sqrt{10} = 1 + \log \sqrt{10} = 1.5$

$\ln 3 + \ln 4 = \ln (3 \cdot 4) = \ln 12 \approx 2.48$

$\ln 4 + \ln 25 = \ln (4 \cdot 25) = \ln 100 \approx 4.605$

$\ln \left(\frac{2}{3}\right) + \ln \left(\frac{3}{4}\right) = \ln \left(\frac{2}{3} \cdot \frac{3}{4}\right) = \ln \left(\frac{2}{4}\right) \approx -0.693$

$\ln (e \cdot \sqrt{e}) = \ln e + \ln \sqrt{e} = 1 + \ln \sqrt{e} = 1.5$

Examples using calculation rule 2

$\log 3 - \log 4 = \log \left(\frac{3}{4}\right) \approx -0.125$

$\log 4 - \log 25 = \log \left(\frac{4}{25}\right) \approx -0.796$

$\log \left(\frac{10}{\sqrt{8}}\right) = \log 10 - \log \sqrt{8} = 1 - \log \sqrt{8} \approx 0.549$

$\ln 3 - \ln 4 = \ln \left(\frac{3}{4}\right) \approx -0.288$

$\ln 4 - \ln 25 = \ln \left(\frac{4}{25}\right) \approx -1.83$

$\ln \left(\frac{e}{\sqrt{8}}\right) = \ln e - \ln \sqrt{8} = 1 - \ln \sqrt{8} \approx -0.0397$

Examples using calculation rule 3

$\log 3^4 = 4 \cdot \log 3 \approx 1.91$

$\log 10^{2x} = 2x \cdot \log 10 = 2x \cdot 1 = 2x$

$1 + \log\sqrt{10} \;=\; 1 + \log 10^{0,5} \;=\; 1 + 0.5\cdot\log 10 \;=\; 1 + 0.5\cdot 1 \;= 1.5$

$\log 10^3 - \log 10^2 \;=\; 3 - 2 \;=\; 1$

$\ln 3^4 \;=\; 4\cdot\ln 3 \;\approx\; 4.39$

$\ln e^{2x} \;=\; 2x\cdot\ln e \;=\; 2x\cdot 1 \;=\; 2x$

$1 + \ln\sqrt{e} \;=\; 1 + \ln e^{0,5} \;=\; 1 + 0,5\cdot\ln e \;=\; 1 + 0.5\cdot 1 \;=\; 1.5$

Examples with equations

$3^x \;=\; 27 \quad\Leftrightarrow\quad \ln 3^x \;=\; \ln 27 \quad\Leftrightarrow$

$x\cdot\ln 3 \;=\; \ln 3^3 \quad\Leftrightarrow\quad x\cdot\ln 3 \;=\; 3\cdot\ln 3 \quad\Leftrightarrow$

$x \;=\; 3$

$1,8^{-2x} \;=\; 4 \quad\Leftrightarrow\quad \ln(1,8^{-2x}) \;=\; \ln 4 \quad\Leftrightarrow$

$-2x\cdot\ln 1.8 \;=\; \ln 4 \quad\Leftrightarrow\quad x \;=\; \frac{\ln 4}{-2\cdot\ln 1.8} \quad\Leftrightarrow$

$x \;\approx\; -1.18$

We will isolate time, t, in the formula for exponential growth

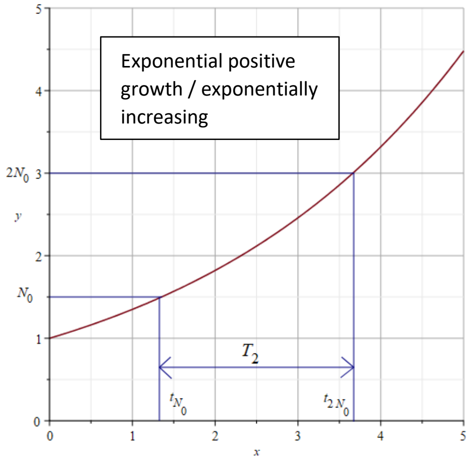
$y \;=\; b\cdot e^{kt} \quad\Leftrightarrow\quad e^{kt} \;=\; \left(\frac{y}{b}\right) \quad\Leftrightarrow$

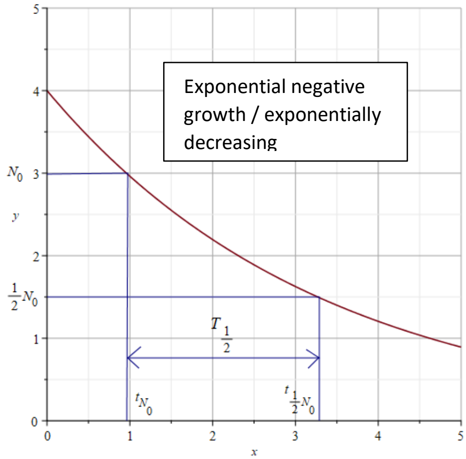
$\ln e^{kt} \;=\; \ln\left(\frac{y}{b}\right) \quad\Leftrightarrow\quad kt \;=\; \ln\left(\frac{y}{b}\right) \quad\Leftrightarrow$

$t \;=\; \frac{1}{k}\cdot\ln\left(\frac{y}{b}\right)$

For instance, that might be the time for growth of a certain bacteria.

More theory





Both functions shown, have the equation $f(x) = b \cdot c^x = b \cdot a^{kx}$

With exponential functions, it is often nice to know when things are doubled or halved. Often, time is on the first axis, so the doubling constant is denoted T_2 , and the halving constant is denoted $T_{½}$, and we can derive formulas for their calculation. On the second axis we here have N for number. The start condition is

$f(t_2) = 2 \cdot f(t)$	=>	$b \cdot c^{t_2} = 2 \cdot b \cdot c^t$	⇔
$c^{t_2} = 2 \cdot c^t$	⇔	$c^{t_2 - t} = 2$	⇔
$ln\, c^{t_2 - t} = ln\, 2$	⇔	$(t_2 - t) \ln c = \ln 2$	⇔
$t_2 - t = \frac{\ln 2}{\ln c}$	⇔	$T_2 = \frac{\ln 2}{\ln c} = \frac{\ln 2}{\ln a^k}$	and
$f(t_{½}) = \frac{1}{2} \cdot f(t)$	=>	$b \cdot c^{t_2} = \frac{1}{2} \cdot b \cdot c^t$	⇔
$c^{t_2} = \frac{1}{2} \cdot c^t$	⇔	$c^{t_2 - t} = \frac{1}{2}$	⇔
$ln\, c^{t_2 - t} = ln\, \frac{1}{2}$	⇔	$(t_2 - t) \ln c = \ln \frac{1}{2}$	⇔
$t_2 - t = \frac{\ln \frac{1}{2}}{\ln c}$	⇔	$T_{½} = \frac{\ln \frac{1}{2}}{\ln c} = \frac{\ln \frac{1}{2}}{\ln a^k}$	

Other functions

Here we will briefly discuss some more rare functions.

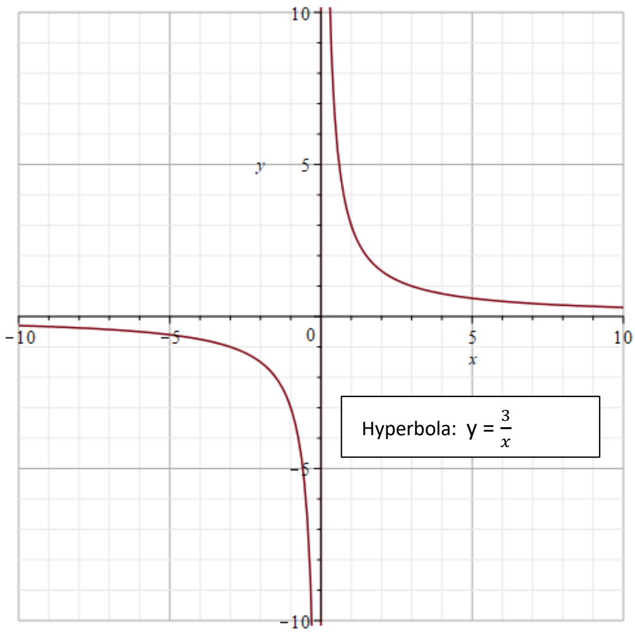
The hyperbola

The hyperbola (Greek: exaggeration) has the equation

$$x \cdot y = k \qquad \Leftrightarrow \qquad y = \frac{k}{x} \qquad \text{or} \qquad x = \frac{k}{y}$$

where k is a constant.

It is seen that neither x nor y can be 0 - the curve cannot pass x = 0 or y = 0 , so the x-axis and the y-axis become asymptotes (which means: no coincidence) to the function (curve). The curve still approaches the axis, but will never reach it.



We meet the hyperbola in Boyle-Mariotte's law (physics) in the form

$p \cdot V = k$ only relevant in the first quadrant

which stands for

pressure · volume = constant

valid at constant temperature.

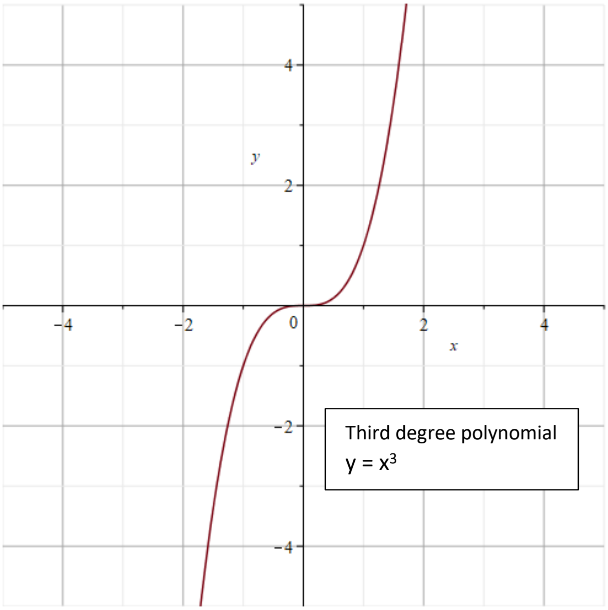
Since neither the absolute pressure nor the volume can be negative the curve is only relevant in the first quadrant.

In techniques, the hyperbola shape applies for special grinded magnetic sensors - and more.

The third degree polynomial

The third degree polynomial comes in many versions, but the basic function is

$$y = x^3$$

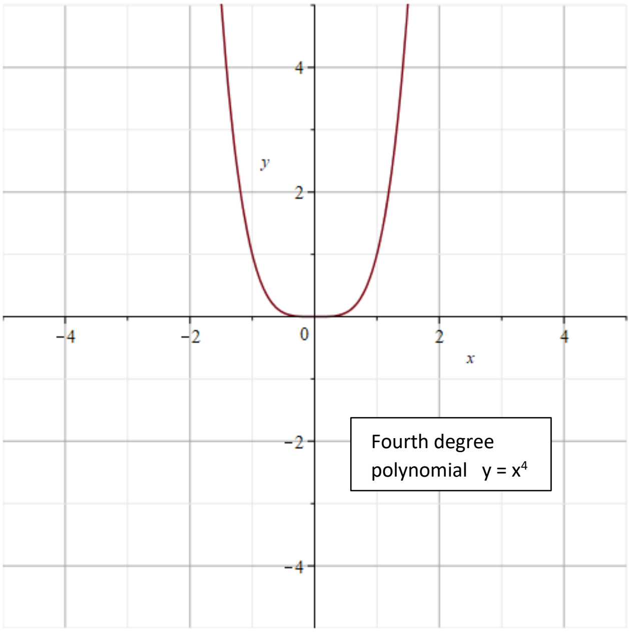


A few laws of nature are third degree equations. For instance Kepler's 3. Law which states that the period, squared, is proportional to its average distance to the sun, to the power of three.

$T^2 = k \cdot r^3$ only relevant in the first quadrant

The fourth degree polynomial

$y = x^4$



A fourth degree polynomial is rare. An example in physics is Stefan-Boltzmann's Law of radiation which states that the radiation intensity from a black body equals a constant times the temperature (in Kelvin) to the power of four:

$I = k \cdot T^4$ only relevant in the first quadrant

Polynomial fraction function

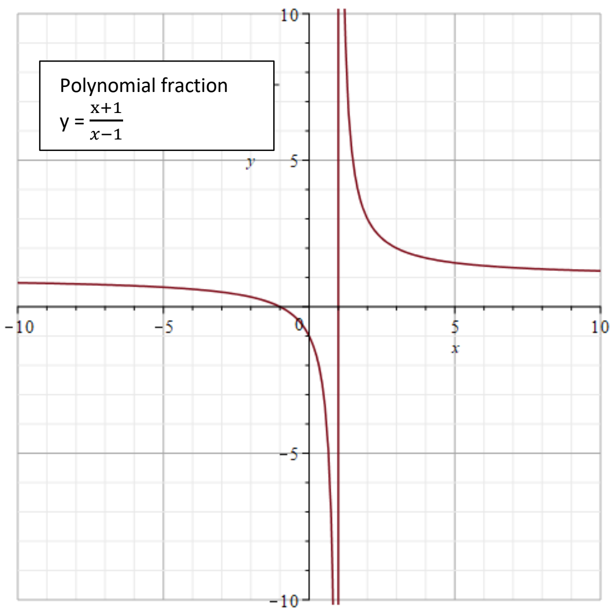
Some functions are so rare that you may not meet them. Yet, they may possess interesting characteristics, which calls for a brief remark. For instance this polynomial fraction function:

$$y = \frac{x+1}{x-1}$$

The denominator cannot be zero, so x cannot be 1. Therefore the curve cannot pass $x = 1$ which consequently becomes asymptote.

$y = 0$ must be if $x = -1$ since the numerator thus will be 0.

Thus, some things we are able to see in advance, - and displayed:



It is observed, that $x = 1$ is a vertical asymptote. Furthermore, we have a suspicion that $y = 1$ is a horizontal asymptote. We will test that by inserting $y = 1$ in the equation:

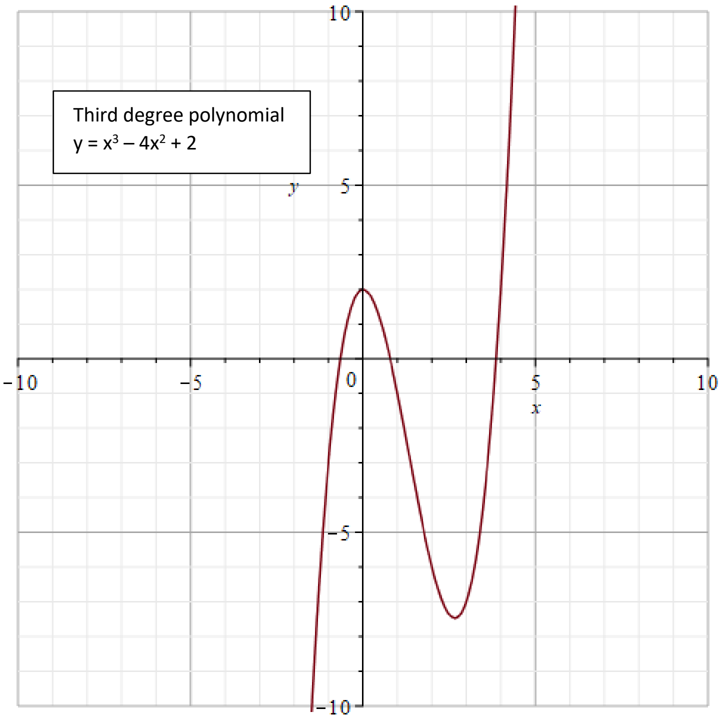
$$y = 1 \quad \Rightarrow \quad x - 1 = x + 1 \quad \Leftrightarrow \quad 0 = 2$$

which is false, and therefore y cannot be 1. This means that the curve cannot pass $y = 1$, which therefore is a horizontal asymptote.

An example of a special third degree polynomial

Finally, a third degree polynomial, which is interesting because it has a local maximum, and a local minimum:

$y = x^3 - 4x^2 + 2$

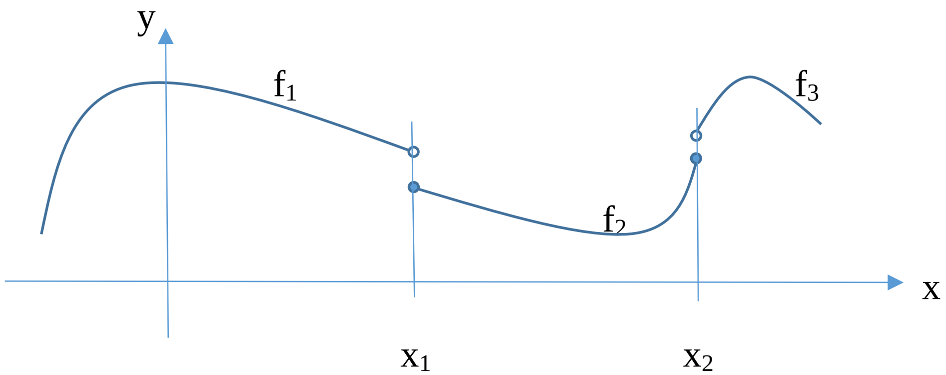


We can easily read the local maximum point: (0, 2)

The local minimum can only be read approximately.

Later, in differential calculation, we shall see how these points may be calculated precisely.

Partly defined functions



Then function shown is in three parts, where each part is defined in an interval

$f_1(x)$ = function equation in the interval: $]-\infty\,;\,x_1[$

$f_2(x)$ = function equation in the interval: $[x_1\,;\,x_2\,]$

$f_3(x)$ = function equation in the interval: $]x_2\,;\,\infty[$

The parts also may have more different names like f, g and h.

Part 3. Differentiation and Integration

Introduction

Now we are going to round the sharp, but also very beautiful, corner of mathematics. We are namely going to investigate quantities that change, as well as how they change. It leads us to new ways of calculation. Previously, we expanded from the four basic arithmetic operations to the trigonometric functions (sin, cos, tan) and the logarithms. Now we expand even further, and the only way to understand this new way of calculation is by understanding the proofs. Differential and integral calculus must be understood through proofs.

First, however, a little mathematical philosophy and history. The ancient Greeks found that not everything proves. We need a basic, which they called axioms - which means basic. From basics and onwards, we have to prove that what we do is correct.

A point has no extent. Does it exist then? Yes, is the answer, and thus we have an axiom. If a point has no extent, there must be an infinite number of points in an area? Yes, - this is another axiom. If the area is bigger, will it contain even more points? Yes. Can something infinite be bigger than some other thing which is also infinite? Yes.

A line has no width, nor does a parabola, a circle or any other curve. Here we have a significant difference between mathematics and other sciences for instance physics. Imagine a spherical ball lying on a plane. In mathematics, there is only contact in one point. In physics, that would produce an infinite surface pressure, which clearly is not so, - no materials could take that. The physical fact is, that both the round ball and the plane will become deformed and form a contact area - not just a point - and thereby form a surface pressure, which can be calculated and measured.

The ancient Greeks were probably not the first to consider these subjects, but we know, that they thought about it. Can we talk about the velocity in a point or at a certain point in time? Can we talk about the reaction speed of a chemical reaction at a certain point in time? Or for biological growth? And, can we make calculations of it?

The answer is yes and yes. The ancient Greeks did not succeed in finding the mathematical basis. It was not until the sixteen hundreds they succeeded in making mathematical proofs. It happened almost at the same time for the physicist Isaac Newton and the mathematician Gottfried Leibnitz. As far as we know, they did not know one another at the time, and their approaches were different. Newton needed novel mathematics to solve physical problems, while Leibnitz was a theoretical mathematician. It is all about calculating differences/changes in the near vicinity of a point, which is why it has the name: differential calculus. A more detailed description of the technical term will follow.

We will now do the long haul of proving differential and integral calculus, and see how much it enables us to calculate, which is comprehensive.

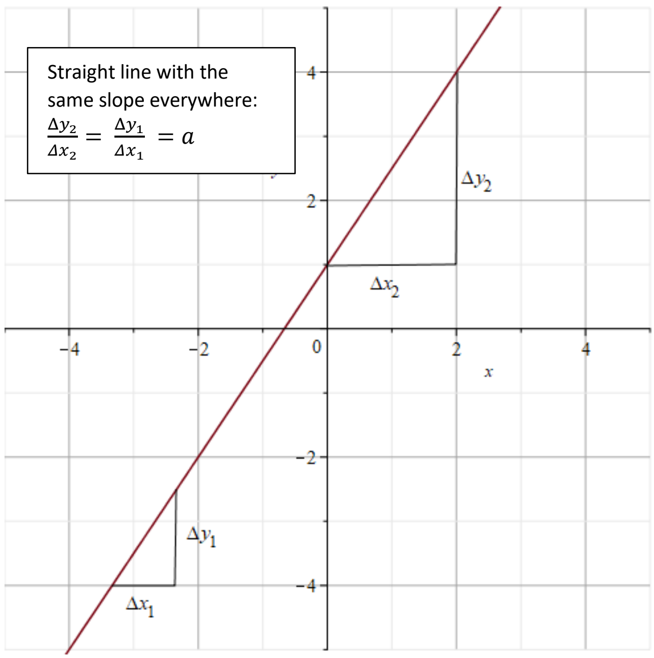
We will not prove everything, but pretty much, though, - the most essential.

Sometimes, differential and integral calculus are called infinitesimal calculus, since we do calculation on infinitesimally small parts when we differentiate, and recollect the infinite number of infinitesimally small parts when we integrate.

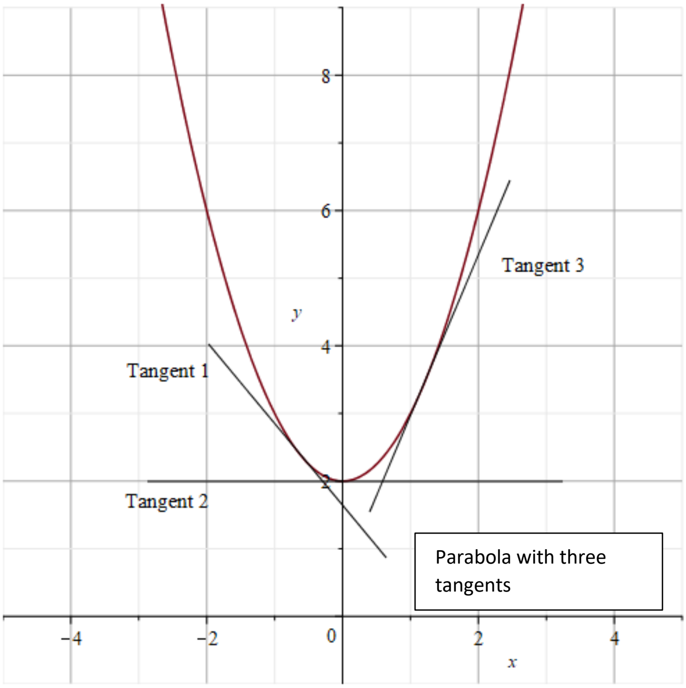
Differential calculus

The slope of a function/curve in a coordinate system describes the change of the quantity in question. If the slope is zero (a horizontal line), there is no change, - we have a constant y-value regardless of the x-value.

Let us start by considering the linear function, where the slope does not change no matter where we are on the line. In other words, the slope is the same for all x regardless of having a small change Δx_1 or a big change Δx_2



All other functions have different slopes in different places, for instance the parabola:

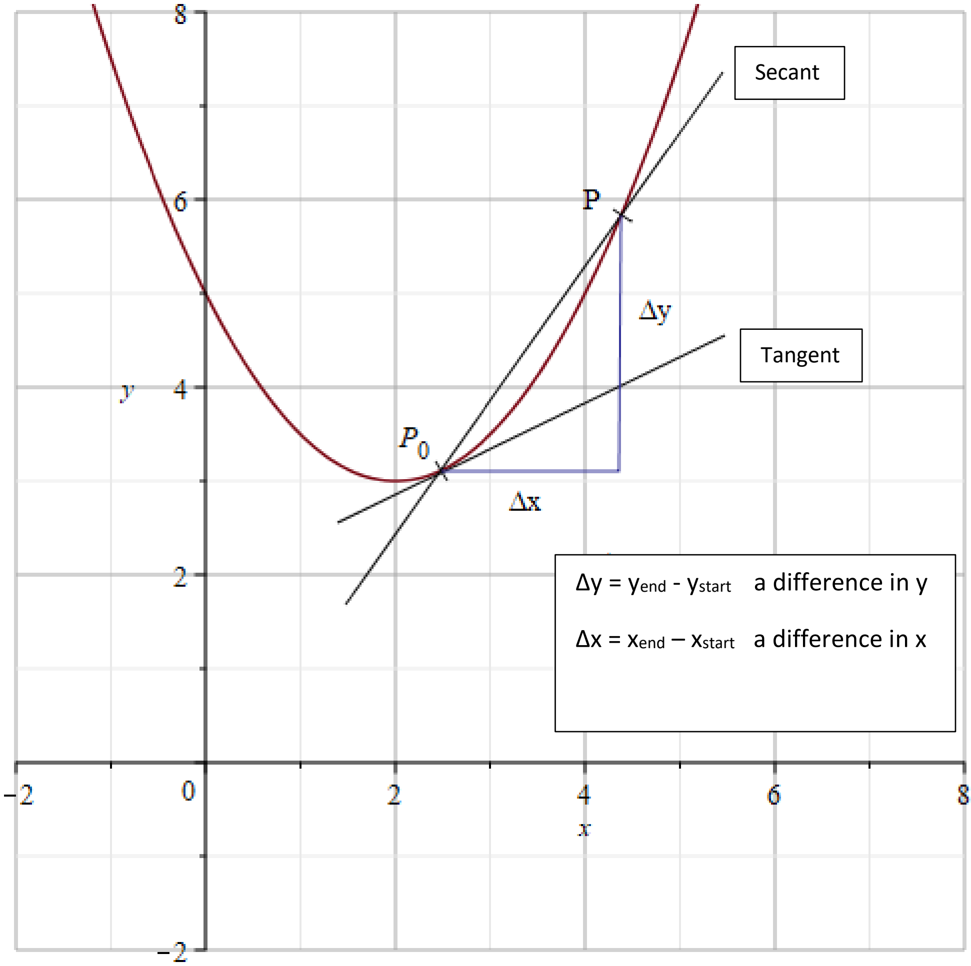


It is observed, that going from left to right, the parabola shown has a negative slope that decreases as we move towards the vertex, zero slope at the vertex, and then a positive and increasing slope as we move further to the right.

The human eye is sharp so we are able to sketch a tangent in a point on the parabola. The tangent only touches the parabola in one point. It is a tangent, and we can read its slope with some uncertainty. Yet, we would like to be precise, so can we find the slope of the tangent by calculation?

If we can determine the slope of the tangent, we can also determine the slope of the curve in that point. They are the same. Oh, but a point has no extent, so how can we talk about the slope of a curve in a point, and how can we calculate it!?

In the next diagram we have a parabola (it could be any curve) with a secant intersecting in points P_0 and P.



Also, there is a blue helping triangle showing the slope of the secant $\frac{\Delta y}{\Delta x}$

Now we imagine point P sliding down the parabola. Thereby the secant will slide with it and have a smaller slope when approaching point P_0. When we have almost reached P_0, the secant almost becomes a tangent in P_0. Δx also gets smaller and is now called δx, while Δy becomes δy.

We have now moved from a macro world where Δx and Δy are big and visible (which is why we use the Greek letter capital Delta, Δ) to a micro world where δx and δy are infinitely small (which is why we use the Greek letter small delta, δ).

Later, d replaced δ, which is the modern small delta in our alphabet. Thus, we have seen this:

- P slides down the parabola and almost becomes coincident with P_0
- The Secant almost becomes tangent
- Δx becomes dx
- Δy becomes dy
- Secant slope $\frac{\Delta y}{\Delta x}$ becomes tangent slope $\frac{dy}{dx}$

Even though P and P_0 are very, very close, they are not the same point. Thus, we have no problem with the axiom stating that a point has no extent, and therefore, *in practice*, we may talk about the slope of a function in a point.

macro $\frac{\Delta y}{\Delta x}$ is called the difference quotient (= difference fraction)

micro $\frac{dy}{dx}$ is called the differential coefficient (= derivative)

In brief:

$\frac{\Delta y}{\Delta x}$ = secant slope = difference quotient

$\frac{dy}{dx}$ = tangent slope = differential coefficient

The next step is to calculate the tangent slope $\frac{dy}{dx}$ for known functions by following the same procedure as just mentioned.

Proofs of differential calculus 1

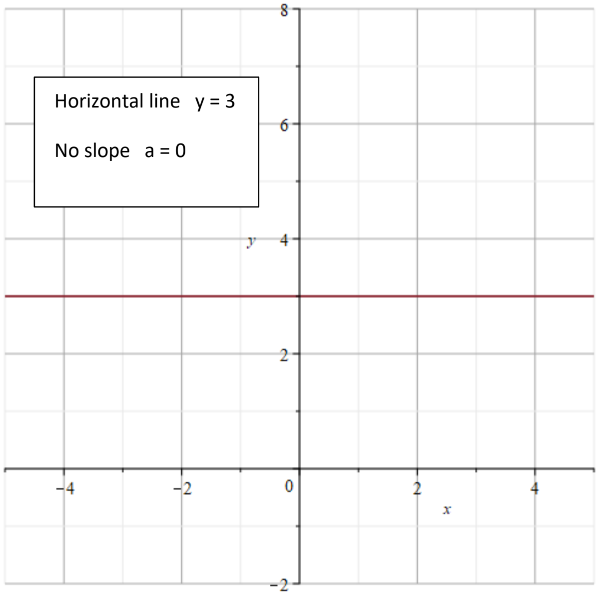
There are some methods. We will use the *three-step-rule*:

1. Calculate Δy
2. Calculate $\frac{\Delta y}{\Delta x}$
3. Let Δx go towards zero to find $\frac{dy}{dx}$

We will begin with a horizontal line, of which we already know that the slope is 0. Thus, we expect to find a differential coefficient of 0.

The horizontal line

$y = b$ or $y = \text{constant}$



1. Calculate Δy here: 0
2. Calculate $\frac{\Delta y}{\Delta x}$ here: 0

3. Let Δx go towards zero to find $\frac{dy}{dx}$ here: 0

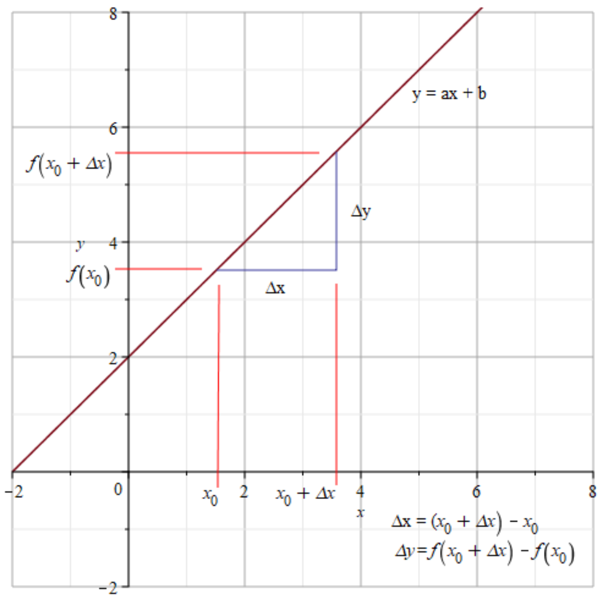
So, when we differentiate a constant we yield 0, i.e. slope = 0

In other words: The difference of a constant is 0, it does not change. As expected.

The straight line

Then we consider the straight line, which we already know has the same slope (namely a), and thus the same differential coefficient everywhere.

$y = ax + b$ or $f(x) = ax + b$



The diagram displays a straight line and a helping triangle with one corner at the x-value x_0. Further to the right the x-value becomes $x_0 + \Delta x$ (since we are the distance Δx further to the right).

The corresponding y-values are now called the function values:
$f(x_0)$ and $f(x_0 + \Delta x)$

Three-step-rule

1. Calculate Δy
   $\Delta y = f(x_0 + \Delta x) - f(x_0)$ =>
   And with our x-values inserted into the line equation $y = ax + b$
   $\Delta y = [a(x_0 + \Delta x) + b] - [ax_0 + b]$ ⇔
   $\Delta y = a \cdot \Delta x$

2. Calculate $\frac{\Delta y}{\Delta x}$

   $$\frac{\Delta y}{\Delta x} = \frac{a \cdot \Delta x}{\Delta x} = a$$

3. Let Δx go towards zero to find $\frac{dy}{dx}$ =>

   Δx was reduced from the equation in point 2, so Δx has no influence on the slope. It becomes
   $\frac{dy}{dx} = \frac{\Delta y}{\Delta x} = a$
   It corresponds nicely with what we already know, namely that a straight line has the same slope everywhere. Here we call it a, at other times we may call it c or k to show that it is a constant.

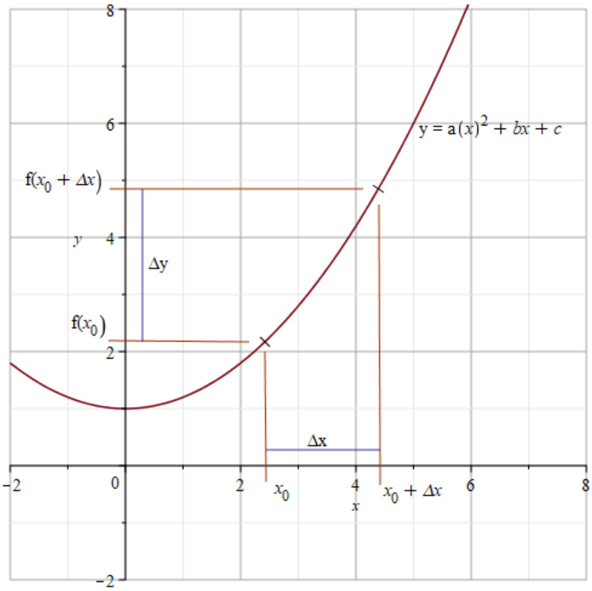
The tangent slope for a straight line - which equals the differential coefficient $\frac{dy}{dx}$ for a straight line - equals the number (the constant) a.

The differential coefficient is also called f´(x). So:

$\frac{dy}{dx} = f'(x) = a$ the reason for also using f´(x) follows later

The parabola

$y = ax^2 + bx + c \quad \Leftrightarrow \quad f(x) = ax^2 + bx + c$



Three-step-rule

1. Calculate Δy

   $\Delta y = f(x_0 + \Delta x) - f(x_0)$

   and with the values of a parabola

   $\Delta y = (a(x_0+\Delta x)^2 + b(x_0+\Delta x)+c) - (ax_0^2+bx_0+c)$

   $\Delta y = (a(x_0^2+(\Delta x)^2 + 2x_0\Delta x)+bx_0+b\Delta x+c) - (ax_0^2+bx_0+c)$

   $\Delta y = ax_0^2 + a(\Delta x)^2 + 2ax_0\Delta x + bx_0 + b\Delta x + c - ax_0^2 - bx_0 - c$

   $\Delta y = a(\Delta x)^2 + 2ax_0\Delta x + b\Delta x$

2. Calculate $\frac{\Delta y}{\Delta x}$

$$\frac{\Delta y}{\Delta x} = \frac{a(\Delta x)(\Delta x) + 2a \cdot x_0 \cdot \Delta x + b\Delta x}{\Delta x} = a\Delta x + 2ax_0 + b$$

3. Let Δx go towards zero to find $\frac{dy}{dx}$ =>

$\frac{dy}{dx} = 2ax + b$ since x_0 changes for x to describe all values of x, not just the one we named x_0

The tangent slope for a parabola, which equals the differential coefficient $\frac{dy}{dx}$, thus becomes an equation:

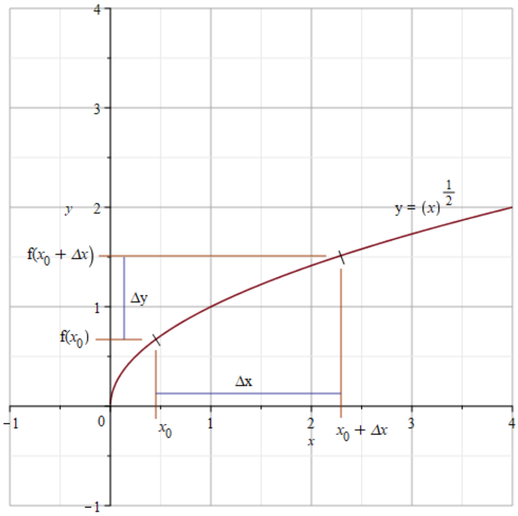
$$\frac{dy}{dx} = f'(x) = 2ax + b$$

a and b are known constants, while x is variable.

The tangent slope depends on x. In other words: the tangent slope depends on where we are on the parabola.

The square root function

$y = \sqrt{x}$ or better $y = x^{½}$ ⇔ $f(x) = x^{½}$



Three-step-rule

1. Calculate Δy
   $\Delta y = f(x_0 + \Delta x) - f(x_0)$
   and with the values of a square root function
   $\Delta y = (x_0 + \Delta x)^{½} - x_0^{½}$

2. Calculate $\frac{\Delta y}{\Delta x}$

$$\Delta y = \frac{(x_0 + \Delta x)^{½} - x_0^{½}}{\Delta x}$$

Numerator and denominator multiplied by $((x_0 + \Delta x)^{½} + x_0^{½})$

$$\Delta y = \frac{((x_0 + \Delta x)^{½} - x_0^{½})}{\Delta x} \cdot \frac{((x_0 + \Delta x)^{½} + x_0^{½})}{((x_0 + \Delta x)^{½} + x_0^{½})}$$

and we use a square rule

$$\Delta y = \frac{(x_0 + \Delta x) - x_0}{\Delta x \cdot ((xo + \Delta x)^{½} + x_0^{½})} \quad \Leftrightarrow$$

$$\Delta y = \frac{\Delta x}{\Delta x \cdot ((x_0 + \Delta x)^{½} + x_0^{½})} \quad \Leftrightarrow$$

$$\Delta y = \frac{1}{(x_0 + \Delta x)^{½} + x_0^{½}}$$

3. Let Δx go towards zero to find $\frac{dy}{dx}$ =>

$$\frac{dy}{dx} = f'(x) = \frac{1}{2\,x^{½}} = \frac{1}{2} x^{-½}$$

x_0 changed to x

The tangent slope depends on x. In other words: the tangent slope depends on where we are on the curve.

Polynomials

$y = x^n \quad \Leftrightarrow \quad f(x) = x^n$ n is also called a

We have just seen the differential coefficient of two polynomials: the parabola and the square root function. What about the other polynomials (x^3, x^4, $x^{2,3}$,…) ?

If we simplify to $y = x^2$ the differential coefficient will be:

$$y = x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 2x^1 = 2x$$

and for the square root function

$$y = x^{½} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}x^{-½}$$

In practice these two functions are differentiated by ”putting the exponent in front as a factor and let the exponent drop 1”. It is easy to see for the parabola: ”2 put in front, and let the exponent drop: 2 - 1 = 1”. For the square root function: ”½ put in front, and the exponent becomes ½ - 1 = -½ ”.

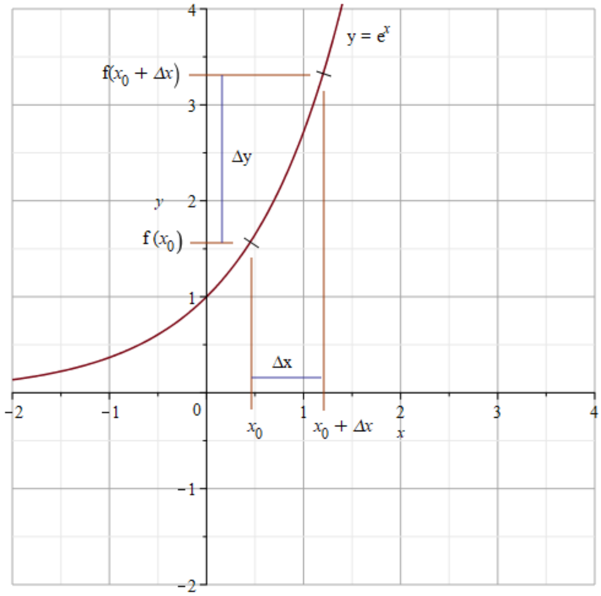
All polynomials are differentiated the same way:

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = n \cdot x^{n-1}$$ the power rule

This we will not prove.

The natural exponential function

$y = e^x \quad \Leftrightarrow \quad f(x) = e^x$



Three-step-rule

1. Calculate Δy

   $\Delta y \; = \; f(x_0 + \Delta x) - f(x_0)$

   and with the values of the function

   $y \; = \; e^{xo+\Delta x} - e^{xo}$

2. Calculate $\frac{\Delta y}{\Delta x}$

   $$\frac{\Delta y}{\Delta x} = \frac{e^{xo+\Delta x} - e^{xo}}{\Delta x} = e^{xo} \cdot \frac{e^{\Delta x} - 1}{\Delta x}$$

3. Let Δx go towards zero to find $\frac{dy}{dx}$

   In the expression we found in point 2, we see that e^{xo} does not change if Δx goes towards 0.

If we look at the fraction $\frac{e^{\Delta x}-1}{\Delta x}$ and let Δx go towards 0, $e^{\Delta x}$ will go towards 1, and thus the numerator will go towards 0.
The denominator will also go towards 0.
However, we cannot see which value the whole fraction goes towards. We need more information:
We get that from the very definition of the function $y = e^x$ where:

$x_0 = 0$ inserted into $e^{xo} \cdot \frac{e^{\Delta x}-1}{\Delta x}$ is a slope of 1:

$x_0 = 0 \quad => \quad 1 \cdot \frac{e^{\Delta x}-1}{\Delta x} = 1$

That can only happen if the fraction goes towards 1.

Combined $e^{xo} \cdot \frac{e^{\Delta x}-1}{\Delta x}$ goes towards $e^{xo} \cdot 1$ when Δx goes towards 0 =>

$\frac{dy}{dx} = f'(x) = e^x$ $\qquad$ x_0 changed to x

e is Euler's number (the base number for the natural logarithm) which is known, while x is variable.

The tangent slope depends on x. In other words: the tangent slope depends on where we are on the curve.

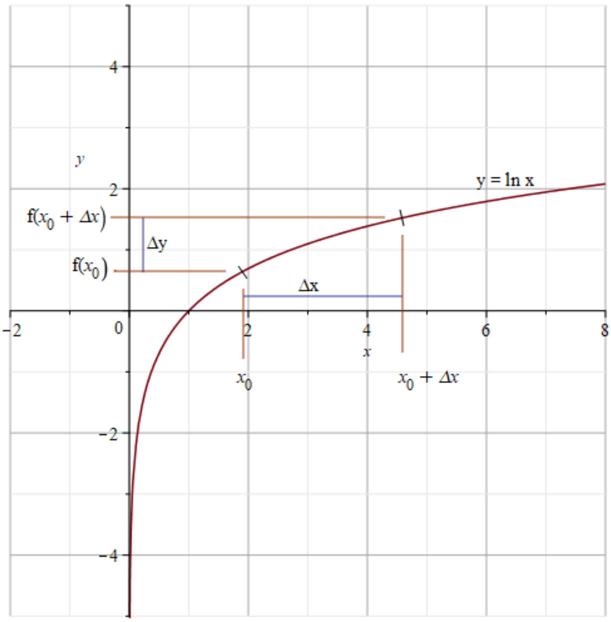
Please note that the tangent slope of the function e^x is e^x

$f(x) = e^x$ $\qquad$ and $\qquad$ tangent slope $= \frac{dy}{dx} = f'(x) = e^x$

No other function has this characteristic.

The natural logarithm

$y = \ln x$ $\qquad$ or $\qquad$ $f(x) = \ln x$



Three-step-rule

1. Calculate Δy

   $\Delta y \;=\; f(x_0 + \Delta x) - f(x_0)$

   and with the values of the function

   $\ln(x_0 + \Delta x) - \ln x_0 \;=\; \ln\left(\frac{x_0+\Delta x}{x}\right)$

2. Calculate $\frac{\Delta y}{\Delta x}$

   $$\frac{\Delta y}{\Delta x} = \frac{\ln\left(\frac{x_0+\Delta x}{x_0}\right)}{\Delta x} = \frac{\ln\left(1+\frac{\Delta x}{x_0}\right)}{\Delta x}$$

   Here we must separate the fraction $\left(\frac{\Delta x}{x_0}\right)$ to see, what happens when we let Δx go towards 0. We do so by calling the fraction k:

   $\frac{\Delta x}{x_0} = k \qquad \Leftrightarrow \qquad \Delta x = kx_0 \qquad =>$

$$\frac{\Delta y}{\Delta x} = \frac{\ln(1+k)}{kx_0} = \frac{1}{x_0} \cdot \frac{\ln(1+k)}{k}$$

3. Let Δx go towards zero to find $\frac{dy}{dx}$
   Now we see that when Δx goes towards 0, k will also go towards 0. This means that both numerator and denominator in $\frac{\ln(1+k)}{k}$ go towards 0, but we cannot see, what value the fraction goes towards, so we need more information:
   We get this by remembering that the ln-function is the inverse of the e^x-function, and thus has the slope 1 for $x_0 = 1$:
   $x_0 = 1$ inserted into $\frac{1}{x_0} \cdot \frac{\ln(1+k)}{k}$ is a slope of 1:
   $x_0 = 1 \quad => \quad \frac{1}{1} \cdot \frac{\ln(1+k)}{k} = 1$
   This can only happen if the fraction goes towards 1.
   Combined $\frac{1}{x_0} \cdot \frac{\ln(1+k)}{k}$ goes towards $\frac{1}{x_0} \cdot 1$ when k and thus Δx goes towards 0 =>

   $\frac{dy}{dx} = f'(x) = \frac{1}{x}$ $\quad\quad$ x_0 changes to x

The tangent slope depends on x. In other words: the tangent slope depends on where we are on the curve.

Generally, another way to write step 3 is:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

lim stands for limes, which is Latin and means limit. Therefore, it says:

The limit value of the difference quotient $\frac{\Delta y}{\Delta x}$ when Δx goes towards 0 is the differential coefficient $\frac{dy}{dx}$

Notations

A differential coefficient may be written in many ways. One way is preferable in some cases, another is preferable in other cases.

The core of it is:

$$\frac{dy}{dx} = \textit{differential coefficient} = \textit{equation for the tangent slope}$$

and then all the other notations:

A function is often called y or y(x), or f(x), or just f. Therefore, the differential coefficient is often called y´, or f´(x), or f´.

If we only use y or f, it is understood that we know what is the name of the unknown - often x. The variable may, of course, be something else, like for instance t for time, and the function may be called anything other than f.

In CAS we often have $\frac{\mathrm{d}}{dx}y$ or $\frac{d}{dx}f(x)$ where the function/equation also may be inserted, for instance $\frac{d}{dx}(x^2+x)$.

Combined we have

$$\frac{dy}{dx} = \frac{\mathrm{d}}{dx}y = \frac{d}{dx}f(x) = y' = f'(x) = f'$$

They all express the same, namely the differential coefficient, which means the equation for the tangent slope, - which inform us about how the function changes.

In words

Differential coefficient = the first derivative

(as we shall see later, we can differentiate one more time and obtain the second derivative).

Differentiation and the four basic arithmetic operations

Maybe we have a function in parts combined by the four basic arithmetic operations (sum, difference, product, division). How do we find the equation for the slope of the function, i.e. the differential coefficient, in such a case?

Here the whole function is called y or f, while the parts of a function are called u and v, then mistakes should be avoided.

Sum (addition).

$y(x) = u(x) + v(x)$ ⇔ or brief

$y = u + v$ => is differentiated to

$y' = (u + v)' = u' + v'$

The function is differentiated part by part

$y = u + v$

if x changes by Δx, the whole function y will have a change of Δy, while the parts of the functions will have a change: Δu and Δv =>

$y + \Delta y = (u + \Delta u) + (v + \Delta v)$ ⇔

$u + v + \Delta y = u + \Delta u + v + \Delta v$ ⇔

$\Delta y = \Delta u + \Delta v$ => from macro to micro

$dy = du + dv$ ⇔ divided by dx

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ⇔ or

$y' = u' + v'$ ⇔

$y' = (u + v)' = u' + v'$ thus, differentiation part by part

Example

$y \;=\; 3x^2 + \ln x \qquad => \qquad y' = 6x + \frac{1}{x}$

Difference (subtraction).

$y(x) \;=\; u(x) - v(x)$ or brief

$y \;=\; u - v \qquad =>$ differentiated

$y' = (u - v)' = u' - v'$

The function is differentiated part by part.

The proof is similar to the sum proof, only v is minus.

Example

$y \;=\; 3x^2 - \ln x \qquad => \qquad y' = 6x - \frac{1}{x}$

Product (multiplication)

$y \;=\; u \cdot v \qquad =>$ differentiated

$y' = (u{\cdot}v)' = u'{\cdot}v + u{\cdot}v'$ which is called the product formula

Proof:

$y \;=\; u \cdot v$

if x has a change of Δx, the whole function y will have a change of Δy, while the parts of the function will change: Δu and Δv =>

$y + \Delta y \;=\; (u + \Delta u) \cdot (v + \Delta v) \qquad \Leftrightarrow$

$u{\cdot}v + \Delta(u{\cdot}v) \;=\; u{\cdot}v + u{\cdot}\Delta v + \Delta u{\cdot}v + \Delta u{\cdot}\Delta v \qquad \Leftrightarrow$

$\Delta(u{\cdot}v) \;=\; u{\cdot}\Delta v + v{\cdot}\Delta u + \Delta u{\cdot}\Delta v \qquad =>$

since $\Delta u \cdot \Delta v$ is infinitesimal (limitless small) the part may be omitted. So, when we go from macro to micro, we have

$d(u \cdot v) = u \cdot dv + v \cdot du$ => divided by dx

$\frac{d(u \cdot v)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$ =>

$y' = (u \cdot v)' = u \cdot v' + v \cdot u'$ the product formula

Example

$y = 3x^2 \cdot \ln x$ => $y' = 3x^2 \cdot \frac{1}{x} + 6x \cdot \ln x$

Division

$y = \frac{u}{v}$ => differentiated

$y' = \left(\frac{u}{v}\right)' = \frac{u' \cdot v - v' \cdot u}{v^2}$ which is called the quotient formula

Proof:

$y = \frac{u}{v}$

if x has a change of Δx, the whole function y will have a change of Δy, while the parts of the function will change: Δu and Δv =>

$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$ =>

$\frac{u}{v} + \Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v}$ $\Leftrightarrow$

$\Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v(u + \Delta u)}{v(v + \Delta v)} - \frac{u(v + \Delta v)}{v(v + \Delta v)} = \frac{v \cdot u + v \cdot \Delta u - u \cdot v - u \cdot \Delta v}{v(v + \Delta v)}$ $\Leftrightarrow$

$\Delta\left(\frac{u}{v}\right) = \frac{v \cdot \Delta u - u \cdot \Delta v}{v(v + \Delta v)}$ since $v \cdot \Delta v$ is infinitesimal we have

$\Delta\left(\frac{u}{v}\right) = \frac{v \cdot \Delta u - u \cdot \Delta v}{v^2}$ => from macro to micro

$d\left(\frac{u}{v}\right) = \frac{v \cdot du - u \cdot dv}{v^2}$ ⇔ divided by dx

$$y´ = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$y´ = \left(\frac{u}{v}\right)' = \frac{u´ \cdot v - v´ \cdot u}{v^2}$ the quotient formula

Example

$y = \frac{3x^2}{\ln x}$ => $\frac{6x \cdot \ln x - x^{-1} \cdot 3x^2}{(\ln x)^2}$

Differentiation of composite functions

If x has ”roles” more than the four rules of arithmetic operations, we talk about a composite function.

The easiest way of differentiating a composite function is by the chain rule, which we derive

$\frac{dy}{dx} = \frac{dy \cdot du}{dx \cdot du}$ ⇔ extension by factor *d*u

$\frac{dy}{dx} = \frac{dy \cdot du}{du \cdot dx}$ ⇔

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ the chain rule

Thus we split the function in the two ”roles” x has, differentiate one by one and gather them by multiplication.

Examples

1.

$y = (x^2 + 1)^3$

here we chose to call the ”inner” function u =>

$\frac{du}{dx} = \frac{d(x^2+1)}{dx} = 2x + 0$

and the outer function y =>

$\frac{dy}{du} = \frac{d(u^3)}{du} = 3u^2 = 3(x^2 + 1)^2$

combined

$2x \cdot 3(x^2 + 1)^2$ which is reduced to $6x \cdot (x^2 + 1)^2$

Or briefly:

$y = (x^2 + 1)^3$

diff. inner $2x$

diff. outer $3(x^2 + 1)^2$

combined $2x \cdot 3(x^2 + 1)^2$

2.

$y = \ln (x^2 - x)$ $\Leftrightarrow$

diff. inner $2x - 1$

diff. outer $\frac{1}{x^2 - x}$

combined $\frac{2x-1}{x^2 - x}$

For information, the chain rule may be expanded to for instance

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

which makes it even more useful. However, we do not take it further.

More theory

With respect to the notation in some tables, we consider the composite function once more by using the chain rule and then alter the notation

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad =>$$

$$\frac{dy}{dx} = \frac{d(f(x))}{d(g(x))} \cdot \frac{d(g(x))}{dx} \qquad \Leftrightarrow$$

$$y' = \frac{d}{d(g(x))} f(x) \cdot \frac{d}{dx} g(x) \qquad \Leftrightarrow$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Some tables use f and g for both the whole function as well as for the parts of the function as is the case here.

As we can see, the derivation is a bit troublesome, but we end at the same: we just have to differentiate inner and outer and multiply the two.

Proofs of differential calculation 2

Utilizing the novel formulas just achieved, we can now perform some more proofs.

Differentiation of e^{kx}

$y = e^{kx}$ or $f(x) = e^{kx}$

is differentiated as a composite function:

”inner”, which is kx diff. to k

”outer”, which is the e function diff. to e^{kx}

combined $\frac{dy}{dx} = f'(x) = k \cdot e^{kx}$

The exponential function

$y = a^x$ or $f(x) = a^x$

rearranged $y = (e^{\ln a})^x \Leftrightarrow y = e^{x \cdot \ln a}$

and differentiated as a composite function:

”inner”, which is $x \cdot \ln a$ diff. to $\ln a$

”outer”, which is the e function diff. to $e^{x \cdot \ln a} = a^x$

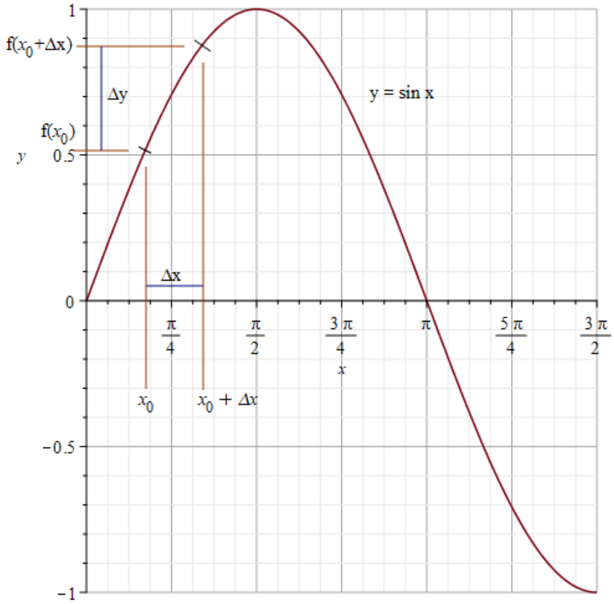
combined $\frac{dy}{dx} = f'(x) = a^x \cdot \ln a$

The sine function

$y = \sin v \Leftrightarrow f(v) = \sin v$ angle v in degrees

or

$y = \sin x \qquad \Leftrightarrow \qquad f(x) = \sin x \qquad$ angle x in radian



Again we will use the three-step-rule:

Three-step-rule

1. Calculate Δy

   $\Delta y = f(x_0 + \Delta x) - f(x_0)$

   and with the values of the function

   $\sin(x_0 + \Delta x) - \sin x_0$

2. Calculate $\frac{\Delta y}{\Delta x}$

   $$\frac{\Delta y}{\Delta x} = \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x}$$

   It can be shown that (we do not show it, we just use it):

   $\sin x - \sin y = 2 \cdot \cos\frac{x+y}{2} \cdot \sin\frac{x-y}{2} \qquad \Rightarrow \qquad$ here

$$\sin(x_0 + \Delta x) - \sin x_0 = 2\cdot\cos\frac{(x_0+\Delta x)+x_0}{2}\cdot\sin\frac{(x_0+\Delta x)-x_0}{2}$$

$$\sin(x_0 + \Delta x) - \sin x_0 = 2\cdot\cos\left(x_0 + \frac{\Delta x}{2}\right)\cdot\sin\frac{\Delta x}{2}$$

inserted

$$\frac{\Delta y}{\Delta x} = \frac{2\cdot\cos\left(x_0+\frac{\Delta x}{2}\right)\cdot\sin\frac{\Delta x}{2}}{\Delta x}$$

We divide by 2 in numerator and denominator

$$\frac{\Delta y}{\Delta x} = \frac{\cos\left(x_0+\frac{\Delta x}{2}\right)\cdot\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

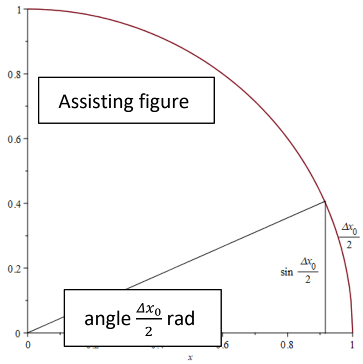
and split in two

$$\frac{\Delta y}{\Delta x} = \cos\left(x_0+\frac{\Delta x}{2}\right)\cdot\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

3. Let Δx go towards 0 to find $\frac{dy}{dx}$

$\cos\left(x_0+\frac{\Delta x}{2}\right)$ goes towards $\cos x_0$

$\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$ goes towards 1, since $\frac{\Delta x}{2}$ in the unit circle is both an angle in radians as well as the angles arc length. When Δx goes towards 0 (the angles gets smaller) sine of the angle and the arc length will go towards the same value. See the figure:



Combined we will go towards cos x_0 =>

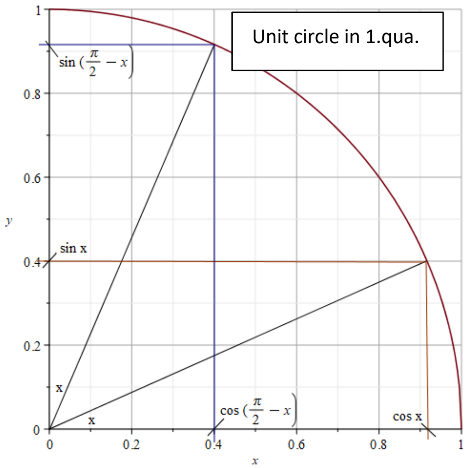
$\frac{dy}{dx}$ = f´(x) = cos x x_0 changes to x

The tangent slope depends on the angle x (here in radians). In other words: The tangent slope depends on where we are on the sine curve.

The cosine function

y = cos x ⇔ f(x) = cos x

sine and cosine are related, and one might be rewritten as the other. Here is an example:



The angle x (here in radians) is marked relative to the x-axis and related to the y-axis. It is seen that

cos x = sin $(\frac{\pi}{2} - x)$

So, instead of differentiating cos x, we can differentiate sin $(\frac{\pi}{2} - x)$. We do it by differentiating outer and inner

$$\frac{dy}{dx} = \cos\left(\frac{\pi}{2} - x\right)\cdot(-1) = -\cos\left(\frac{\pi}{2} - x\right)$$

as seen from the figure equals - sin x =>

$$\frac{dy}{dx} = f'(x) = -\sin x$$

The tangent slope depends on the angle x (here in radians). In other words: The tangent slope depends on where we are on the cosine curve.

The tangent function

$y = \tan x \quad \Leftrightarrow \quad f(x) = \tan x$ angle x in radians

We use the definition of tangent

$$y = \tan x = \frac{\sin x}{\cos x}$$

and the quotient formula

$$\frac{dy}{dx} = f'(x) = \frac{(\sin x)'\cdot\cos x - (\cos x)'\cdot\sin x}{(\cos x)^2} = \frac{\cos x\cdot\cos x - (-\sin x)\cdot\sin x}{(\cos x)^2}$$

$$= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2} \quad \text{or} \quad 1 + (\tan x)^2$$

Thus, there are two similar answers expressed differently:

$$\frac{dy}{dx} = f'(x) = \frac{1}{(\cos x)^2} = 1 + (\tan x)^2$$

Survey

We have now laid the foundation for the differential calculus by proving and deriving a lot. Thus, we also have laid the foundation for the integral calculus, which we will describe later.

The survey is:

function	*derivative* *(differentiation of function)*
y or f(x)	$\frac{dy}{dx}$ or f´(x)
constant (c, k, a, or....)	0
$ax + b$	a
$ax^2 + bx + c$	$2ax + b$
$x^{½}$ or $\sqrt{x}$	$\frac{1}{2}\,x^{-½}$
x^n	$n \cdot x^{n-1}$
e^x	e^x
$\ln x$	$\frac{1}{x} = x^{-1}$
e^{kx}	$k \cdot e^{kx}$
a^x	$a^x \cdot \ln a$
$\sin x$	$\cos x$
$\cos x$	$- \sin x$
$\tan x$	$\frac{1}{(\cos x)^2} = 1 + (\tan x)^2$

$y = u + v \quad => \quad y' = (u + v)' = u' + v'$

$y = u - v \quad => \quad y' = (u - v)' = u' - v'$

$y = u \cdot v \quad => \quad y' = (u \cdot v)' = u' \cdot v + u \cdot v'$

$y = \frac{u}{v} \quad => \quad y' = \left(\frac{u}{v}\right)' = \frac{u' \cdot v - v' \cdot u}{v^2}$

$y = y(u(x)) \quad => \quad y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$y = f(g(x)) \quad => \quad y' = (f(g(x)))' = f'(g(x)) \cdot g'(x)$

The two latter formulas express the same.

Examples

1.

Solutions with keywords. The first examples give an answer in alternative notation:

$f(x) = 2x^2$	=>	$f'(x) = 4x$	power
$f = 2x$	=>	$f' = 2$	power
$y = 2$	=>	$y' = 0$	power
$f(x) = 2a + 7$	=>	$f'(x) = 0$	constants
$y = 2k + 117$	=>	$\frac{dy}{dx} = 0$	constants
$f(x) = 2a + 3b$	=>	$\frac{d}{dx} f(x) = 0$	constants

$f(x) = 4x^3 + 2 \ln x$	=>	$f'(x) = 12x^2 + 2 \cdot \frac{1}{x}$	term by term
$y = 4x^3 + a \cdot \ln x$	=>	$\frac{dy}{dx} = 12x^2 + a\frac{1}{x}$	term by term
$y = 4x^3 + \ln x$	=>	$\frac{d}{dx}y = 12x^2 + \frac{1}{x}$	term by term

$4x^3 - \ln x$	=>	$12x^2 - \frac{1}{x}$	term by term
$4x^3 \cdot \ln x$	=>	$12x^2 \cdot \ln x + 4x^3 \cdot \frac{1}{x}$	product
$\frac{4x^3}{\ln x}$	=>	$\frac{12x^2 \cdot \ln x - 4x^3 \cdot x^{-1}}{(\ln x)^2}$	quotient
$x^{\frac{1}{3}}$	=>	$\frac{1}{3} \cdot x^{-\frac{2}{3}}$	power
$x^{\frac{1}{2}}(x^{\frac{1}{2}} - 4) = x - 4x^{\frac{1}{2}}$	=>	$1 - \frac{1}{2} \cdot 4 \cdot x^{-\frac{1}{2}} = 1 - 2x^{-½}$	term by term
$\frac{x}{x+1}$	=>	$\frac{1(x+1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$	quotient
$(e^x - 2x)^3$	=>	$3(e^x - 2x)^2 \cdot (e^x - 2)$	outer, inner
6^{-x}	=>	$6^{-x} \cdot \ln 6 \cdot (-1)$	outer, inner
$xe^x - 1$ =>	$(1 \cdot e^x + x \cdot e^x) - 0 = e^x(1 + x)$		product

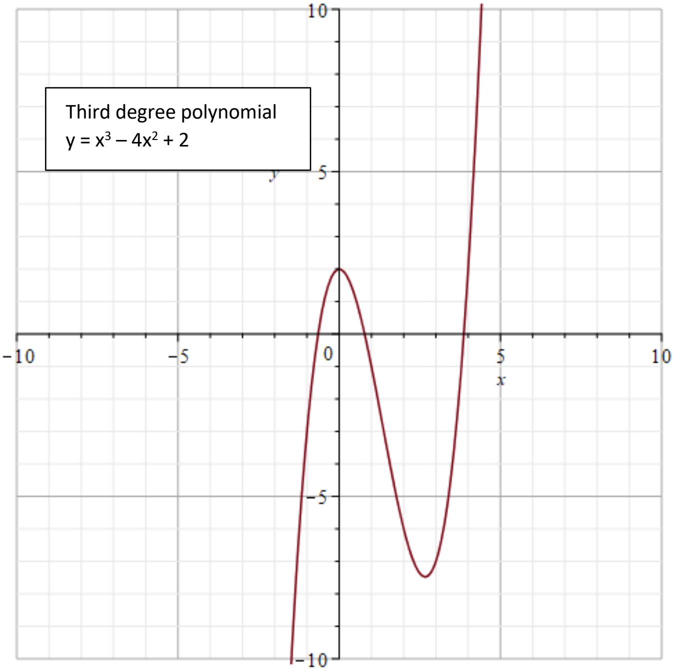
$f(t) = A \cdot \sin(\omega t + \varphi) + k$ =>

$f'(t) = A \cdot \cos(\omega t + \varphi) \cdot \omega + 0$ outer, inner and k to 0

2.

Now we are able to calculate the local maxima and minima (combined called extreme values) etc. in a function previously mentioned.

(An investigation of a function is to find where it is increasing/decreasing, has a maximum/minimum, and maybe asymptotes?)



We do so by finding the places where the tangent slope is zero, which means the places where the differential coefficient is zero.

$y \;=\; x^3 - 4x^2 + 2$ $\qquad\Rightarrow$

$\dfrac{dy}{dx} \;=\; 3x^2 - 2\cdot 4x + 0 \;=\; 0$ $\qquad\Leftrightarrow$

$3x^2 - 8x \;=\; 0$ $\qquad\Leftrightarrow$

$x(3x - 8) \;=\; 0$ $\qquad\Leftrightarrow$

$x_1 \;=\; 0$ and $x_2 \;=\; \dfrac{8}{3}$ and inserted in the function

$y_1 \;=\; 2$ and $y_2 \;=\; \left(\frac{8}{3}\right)^3 - 4\left(\frac{8}{3}\right)^2 + 2 \;=\; -\dfrac{202}{27} \approx -7.48$ $\qquad\Rightarrow$

there are extreme values in points $(0, 2)$ and $\left(\frac{8}{3}, -\frac{202}{27}\right)$

At the curve we see that (0, 2) is a local maximum and $\left(\frac{8}{3}, -\frac{202}{27}\right)$ is a local minimum and that it corresponds nicely with a reading.

If CAS is not available, we must investigate the function before, between, and after the two x-values to decide if it is a minimum or a maximum. We do so by

- insertion of for instance -1 into the differential coefficient =>
  $y = 3\cdot(-1)^2 - 8(-1) = 11$ which is a positive slope showing that the function increases
- insertion of for instance +1 into the differential coefficient =>
  $y = 3\cdot(1)^2 - 8(1) = -5$ which is a negative slope showing that the function decreases
- and insertion of for instance +3 into the differential coefficient =>
  $y = 3\cdot(3)^2 - 8(3) = 3$ which is a positive slope that shows that the function increases again

Therefore, (0, 2) is a local maximum, and $\left(\frac{8}{3}, -\frac{202}{27}\right)$ is a local minimum.

3.

Previously, we saw the parabola

$h(x) = -x^2 - 3$

and we found its vertex: $T\left(\frac{-b}{2a}, \frac{-d}{4a}\right) = T(0, -3)$

We can also find the vertex via the differential coefficient:

At the vertex the tangent slope is 0 (horizontal), thus the differential coefficient is 0. We use this information:

$h'(x) = -2x = 0 \qquad => \qquad x = 0$

which is inserted in the parabola equation to find the y-value

$h(x) = -0^2 - 3 = -3 \qquad => \qquad T(0, -3)$

same answer.

4.

A factory produces a special measuring tool, to be sold at 300 pounds each. The Profit equals Income minus Expenses

$P = I - E$

The market cannot be saturated so I equals the price of one item times the number of items, x

$I = \text{price} \cdot \text{number of sold (= number produced)} = 300 \cdot x$

The expenses divides into fixed costs (mainly new equipment) and variable costs (operation expenses). It is estimated that

$E = F + V = (10\,000) + (11 \cdot x + x^2)$

11x is expenses proportional to the number of produced items, while x^2 eventually becomes significant since the production equipment is worn.

What is the expense per produced item?

When will the profit per produced item be maximum?

When will the production render a deficit?

The expense per item is

$\frac{E}{x} = \frac{10\,000+11x+x^2}{x}$

The profit per produced item is maximum when

$\frac{E}{x}$ = f(x) is minimum, which happens when the slope, i.e. the differential coefficient is zero: $\left(\frac{E}{x}\right)' = 0$ =>

$\left(\frac{E}{x}\right)' = \frac{(11+2x)\cdot x-1\cdot(10\,000+11x+x^2)}{(x)^2} = 0$ ⇔ CAS

x = 100 (or x = -100 which cannot be used)

Thus, the profit per produced item is maximum for item no.100.

Surely, the production gives a deficit at first, and again later when wearing becomes severe. These two points derives from equality between profit per item and expenses per item:

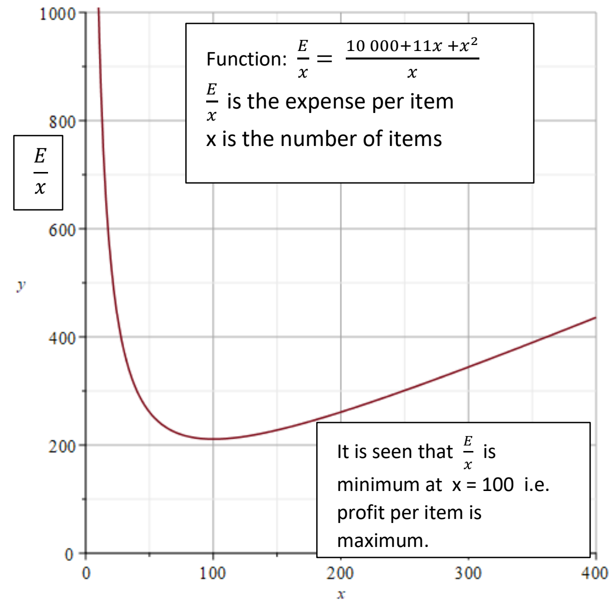
$300 = \frac{E}{x}$ =>

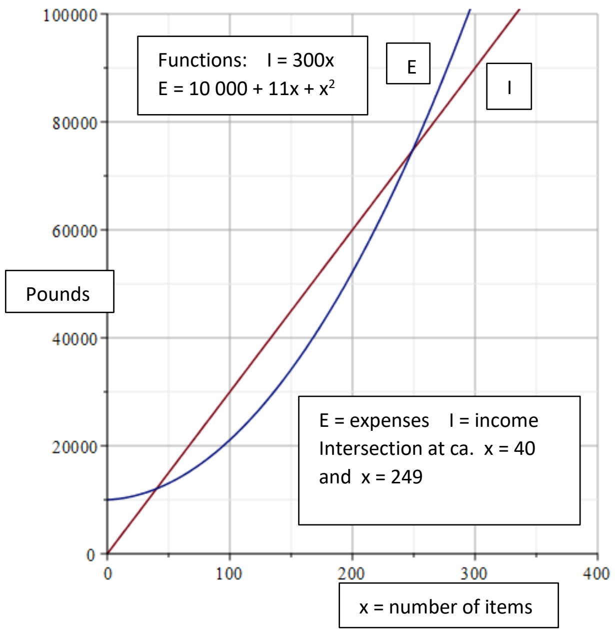
$300 = \frac{E}{x} = \frac{10\,000+11x+x^2}{x}$ ⇔ CAS

x ≈ 40 and x ≈ 249

Thus, deficit until we have produced 40 items, profit until 249 items are produced, and deficit thereafter.

Let us have an overview in diagrams:





Calculation and readings match nicely.

5.

A free fall is linear and with a constant (within limits) acceleration, if we rule out the air resistance. Galilei deduced the following law of nature around year 1600: If we have t for time, s for position (stretch), v for velocity, and g for the acceleration of gravitation, he found the formula

$$s = \frac{1}{2} \cdot g \cdot t^2$$

Approximately 100 years later, when Newton had derived the differential calculus so it was possible to calculate in ”points”, he continued the work of Galilei, and started out by defining:

The momentary velocity: $v = \frac{\text{distance}}{\text{time}} = \frac{ds}{dt}$

and the momentary acceleration: $a = \frac{\text{velocity}}{\text{time}} = \frac{dv}{dt}$

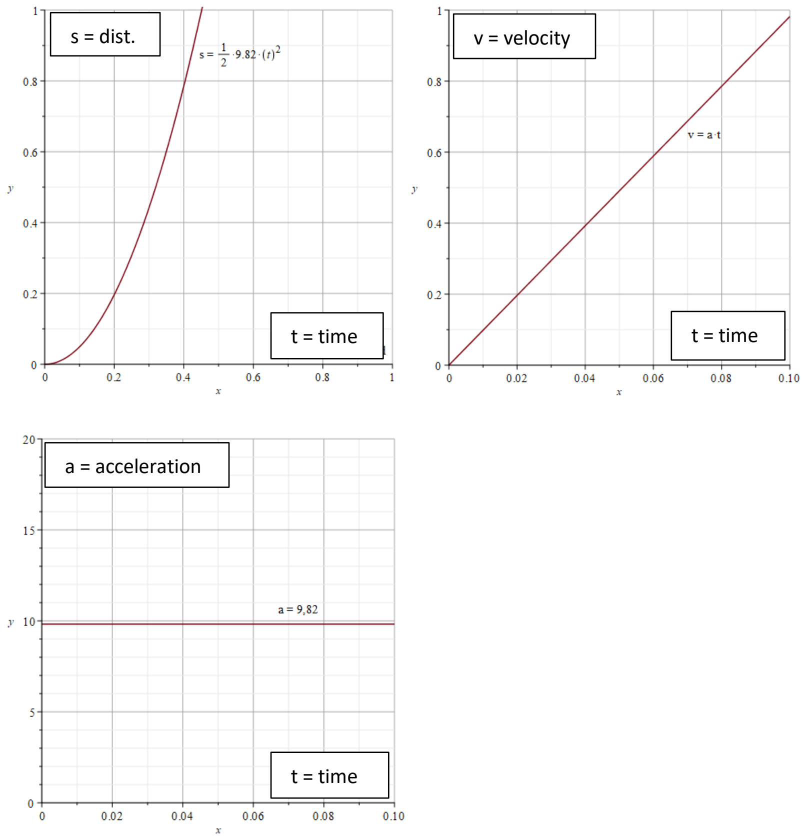
Thus, when the equation for distance is differentiated once with respect to time, we get the velocity, - and when we differentiate the second time with respect to time, we get the acceleration:

$s = \frac{1}{2} \cdot g \cdot t^2$ the equation of a parabola =>

$v = \frac{ds}{dt} = a \cdot t$ the equation of a straight line =>

$a = g$ which is constant (equal to 9.82 m/s^2)

A t,s diagram shows half a parabola, where the tangent slopes show the velocity. A t,v diagram shows a straight line, where the slope is the acceleration. A t,a diagram shows a horizontal line:



The above may also be written this way:

$$v = \frac{\text{distance}}{\text{time}} = \frac{ds}{dt} = s'$$

which is a first order derivative

and

$$a = \frac{\text{velocity}}{\text{time}} = \frac{dv}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt} = \frac{d^2s}{dt^2} = v' = s''$$

which is a second order derivative.

6.

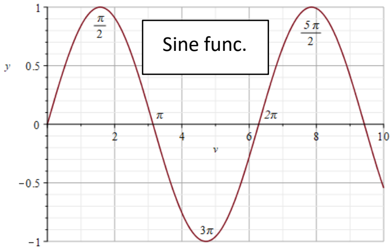
In example 5, the second derivative meant something physical, namely

a = s´´ acceleration = distance diff. twice with respect to time

In other examples, the second derivative just means the tangent slope of the curve of the first derivative. This may be used in an investigation of a function, as we shall see here:

We will consider the sine function

y = sin v v is the angle in radians



Where has the sine curve maximum slope?

(It seems to be at π, 2π, . . etc., but let us calculate precisely):

It has to be at maximum differential coefficient.

The equation for the slope/differential coefficient is

y´ = cos x

which has a maximum, when its own differential coefficient is 0:

y´´ = - sin x = 0

which is for the angles $\pi, 2\pi, 3\pi, \ldots . p \cdot \pi$ where p is a whole number.

It corresponds with what we believe to read from the diagram.

More theory

Is it possible to go on differentiating a third, a fourth,….time?

In principle, yes, if our variable x (or as here, t) is in a power high enough. In the case of the free fall we would have to differentiate a with respect to time in a third differentiation. That would yield 0, and then it is over.

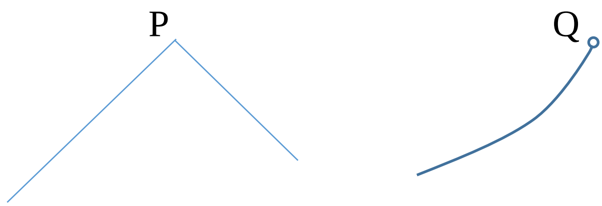
We have previously mentioned that a fifth degree equation is possible in mathematics, but hardly anywhere else. It would be possible to differentiate five times, and each time we find the slope of the curve, but without any other meaning. Math is infinite, but our part of the world is not.

Differentiable - not differentiable

We are able to calculate differential coefficients in ”points” of a curve where it has tangents, and we state, that the function is differentiable.

If we cannot approach the point (here P and Q) at the same curve and from both sides, the curve/function is discontinuous and we cannot determine a tangent. Thus, we cannot calculate the differential quotient either, and the function is not differentiable in these points.

A few examples:



Since we cannot determine the limit value in points P and Q, the functions are not differentiable in P and Q.

Integral calculus

In differentiation calculus we cut into very small pieces to investigate details. In integration calculus, we gather the small pieces again so they make a whole, - going back. So, if we first differentiate a function and then integrate, we will return to the original function. However, there might have been a constant, which disappeared during the differentiation, and consequently will be an unknown when we integrate back.

Thus, we carry out the integration by calculating inversely. All proofs are made during differentiation, now we ”just” have to use the survey inversely.

We may go back from a differentiated function to the function, i.e. from f´ to f, - or we may just integrate a function, i.e. from f to F. F is named the base function (back to basis).

Often within the Natural Sciences we actually know more about the details, than we do about the whole. For example we may observe something changing here and now, but what will it be like over time? Then, we need integration.

It will, however, take a little while until we get to exiting problems like these. First, we must consider how to integrate, which is exiting on its own.

Survey:

derivative *(differentiation of function)*	*function*
$\frac{dy}{dx}$ or f´(x)	y or f(x)

Or:

function	*base function*	
f(x)	F(x)	
0	constant	(often named c or k)
a	ax	
2ax + b	$ax^2 + bx$	
$\frac{1}{2}\, x^{-½}$	$x^{½}$ or $\sqrt{x}$	
$x^{½}$	$\frac{2}{3} \cdot x^{\frac{3}{2}}$	
$n \cdot x^{n-1}$	x^n	or:
x^n	$\frac{1}{n+1} \cdot x^{n+1}$	

”Thus, we increase the exponent by 1 and divide by the new exponent”

$\frac{1}{x} = x^{-1}$	ln |x|	|x| since x may be negative
ln x	x·ln x - x	
e^x	e^x	
e^{kx}	$\frac{1}{k} \cdot e^{kx}$	
a^x	$\frac{1}{\ln a} \cdot a^x$	

cos x	sin x
sin x	- cos x
tan x	-ln |cos x|

Proofs

The two novel functions must be proved.

ln x is proved by differentiating the result (product and term by term):

x·ln x - x diff. => $(1 \cdot \ln x + x \cdot \frac{1}{x}) - (1) = \ln x$

and tan x also proves by diff. of the result (outer, inner):

-ln |cos x| diff. => $-\frac{1}{\cos x} \cdot (-\sin x) = \tan x$

Notations

We write an integral using this sign: $\int$

A stretched S to show that we find the sum, we summate, we gather, we integrate. To integrate means to gather and integration means a gathering. We gather all the very small pieces we made by the derivative.

If our derivative is f´(x) we may write:

$$\frac{dy}{dx} = f'(x) \qquad \Leftrightarrow$$

$$dy = f'(x) \cdot dx$$

Now we want to return to the whole. We do so by gathering all the small dy pieces on the left side - and by gathering all the small pieces dx times f´(x) on the right side:

$$\int dy = \int f'(x) \cdot dx$$

On the left side it is simple: First we cut macro y, into micro dy, and then we reassemble them to y:

$$y = \int f'(x) \cdot dx \qquad \text{or} \qquad f(x) = \int f'(x) \cdot dx$$

This is how we write a normal integral, called an indeterminate integral, which yields the complete solution.

Here we must use the already proved calculation rules from the survey (and from mathematical tables) to calculate the right side.

Examples

1.

We found the derivative of the function

$f(x) = x^2 + x + 3 \quad => \quad f'(x) = 2x + 1 + 0$

in order to look at details, which are slopes in points on the curve.

Now we go back to the whole. We do so by gathering, - by integration:

$y = \int f'(x) \cdot dx$

$y = \int (2x + 1)\, dx$ usually we omit the multiplication dot =>

$y = x^2 + x + k$

If we will find k, we need more information on the function.

Instead of y we might have written f(x).

2.

$f(x) = x^2 + x^3 \quad =>$

$F(x) = \int f(x)\, dx \quad =>$

$F(x) = \int (x^2 + x^3)\, dx \quad \Leftrightarrow$

$F(x) = \frac{x^3}{3} + \frac{x^4}{4} + k$

3.

We will do some more examples. Actually, we could just go back to *Example 1* in the derivative calculus and turn around the implication sign from => to <= and thus go from differentiation to integration by adding a constant:

$f(x) = 2x^2 + k \quad <= \quad f'(x) = 4x$

Anyway, we will solve some more problems by writing the first answers in different ways/notations:

$\frac{dy}{dx} = 2x + 3$	=>	$y = x^2 + 3x + k$
$y' = 3x^2 - 2x + 2$	=>	$y = x^3 - x^2 + 2x + k$
$\frac{d}{dx}y = -2x + \frac{3}{x^2} = -2x + 3x^{-2}$	=>	$y = -x^2 - 3x^{-1} + k = -x^2 - \frac{3}{x} + k$
$\frac{d}{dx}f(x) = 5x^{-6}$	=>	$f(x) = -x^{-5} + k$
$f'(x) = 4x^{-½}$	=>	$f(x) = 8x^{½} + k$
$f' = 2$	=>	$f = 2x + k$

$f(x) = 2\pi$	=>	$F(x) = 2\pi x + k$	π is a number
$f = e^x$	=>	$F = e^x + k$	

$e^7 \cdot e^{-x}$	=>	$e^7 \cdot e^{-x} \cdot (-1) + k$	e^7 is a number
$6^{-2x} = (6^{-2})^x$	=>	$\frac{6^{-2x}}{\ln(6^{-2})} + k$	
$\frac{x^4}{4}$	=>	$\frac{x^5}{4 \cdot 5} + k = \frac{x^5}{20} + k$	
$x^{\frac{3}{4}}$	=>	$\frac{4}{7} \cdot x^{\frac{7}{4}} + k$	
$\ln x$	=>	$(x \cdot \ln x - x) + k$	
$\frac{1}{x}$	=>	$\ln\lvert x\rvert + k$	

Integration and the four basic arithmetic operations

Sum

Just like finding derivatives, we can integrate functions part by part and add them, or we can do the integration under the same integration sign:

$\int u(x)\,dx + \int v(x)\,dx = \int (u(x) + v(x))\,dx$

We prove this by calculating the derivative on the whole of the left side part by part

$\left(\int u(x)\,dx + \int v(x)\,dx\right)' = \left(\int u(x)\,dx\right)' + \left(\int v(x)\,dx\right)' = u(x) + v(x)$

And the derivative of the whole of the right side

$\left(\int (u(x) + v(x))\,dx\right)' = u(x) + v(x)$ gives the same (the right sides are alike).

Difference

This time we write in brief, implied that u and v are functions of x

$\int u\,dx - \int v\,dx = \int (u - v)\,dx$

We prove this by calculating the derivative of the whole of the left side part by part

$\left(\int u\,dx - \int v\,dx\right)' = \left(\int u\,dx\right)' - \left(\int v\,dx\right)' = u - v$

And the derivative of the whole of the right side

$\left(\int (u - v)\,dx\right)' = u - v$ gives the same

Product

We may multiply by a constant inside or outside an integral, thus we may move it. This is because a constant does not change whether we go from macro to micro, or back to macro. It is a constant.

To avoid confusion, with the integration constant k above, we call the new constant c.

$\int c \cdot u(x)\, dx = c \cdot \int u(x)\, dx$

again we prove it by differentiation of the whole of the left side

$(\int c \cdot u(x)\, dx)' = c \cdot u(x)$

and by diff. of the whole of the right side

$(c \cdot \int u(x)\, dx)' = c \cdot u(x)$ gives the same

Example 1

$\int (\frac{1}{x} + \ln x)\, dx = \int \frac{1}{x}\, dx + \int \ln x\, dx = (\ln|x|) + (x \cdot \ln x - x) + k$

$\int (\ln x -117)\, dx = \int \ln x\, dx - \int 117\, dx = (x \cdot \ln x - x) - (117x) + k$

$\int c \cdot x\, dx = c \cdot \int x\, dx = c \cdot \frac{1}{2} \cdot x^2 + k = c_1 \cdot x^2 + k$

Since c is unknown anyhow, we may put c and $\frac{1}{2}$ together in a new constant we call c_1

$\int a \cdot b \cdot x\, dx = ab \int x\, dx = ab \cdot \frac{1}{2} x^2 + k = cx^2 + k$

again the constants are put together as c

Integration by substitution

Some integrals are difficult to solve. We will therefore show some smart methods, which may help us. The first is integration by substitution. As we have seen before, in mathematics, it is allowed to pick out some sizes or parts and call them something else - we substitute. Then we continue calculating with the novelty and usually (but not always) finish by substituting back. It is a fine method, when x has more ”roles”.

Examples

1.

$\int (4x - 2)^{½} \, dx$

we chose to substitute 4x - 2. We call it t

$\int t^{½} \, dx$ where $t = 4x - 2$

We cannot gather dx in a t way, so dx must change for dt. We do so by

$t = 4x - 2 \quad \Rightarrow \quad \frac{dt}{dx} = 4 \quad \Leftrightarrow \quad dx = \frac{dt}{4}$

which is inserted

$\int t^{½} \frac{dt}{4} \quad = \quad \int t^{½} \cdot \frac{1}{4} \cdot dt$

$\frac{1}{4}$ is a constant and is moved “outside”

$\frac{1}{4} \int t^{½} \, dt$

Now we can gather dt in a t way, thus, we can integrate

$\frac{1}{4} \cdot \frac{2}{3} t^{\frac{3}{2}} + k_t$

and, we substitute back to x

$\frac{1}{4} \cdot \frac{2}{3}(4x-2)^{\frac{3}{2}} + k_x$ which is the answer (may be reduced)

The integration constant k_t belongs to the t-expression and changes name to k_x in the x-expression.

And briefly:

$\int (4x - 2)^{½}\, dx$	choice:	$t = 4x - 2$	=>
		$\frac{dt}{dx} = 4$	⇔
		$dx = \frac{dt}{4}$	
$\int t^{½} \frac{dt}{4}$	=		
$\frac{1}{4} \int t^{½}\, dt$	=		
$\frac{1}{4} \cdot \frac{2}{3} t^{\frac{3}{2}} + k_t$	=		
$\frac{1}{4} \cdot \frac{2}{3}(4x-2)^{\frac{3}{2}} + k_x$		which is the answer (may be reduced)	

2.

$\int \frac{2x}{x^2-3}\, dx$	choice:	$t = x^2 - 3$	=>
		$\frac{dt}{dx} = 2x$	⇔
		$dx = \frac{dt}{2x}$	
$\int \frac{2x}{t} \cdot \frac{dt}{2x}$	=		

$\int \frac{1}{t} \cdot dt$ =

$\ln |t| + k_t$ =

$\ln |x^2 - 3| + k_x$ which is the answer

There are no rules for what we may call t, and we must prepare for making another choice. The author suggests to choose “the innermost” and/or “the most complicated” - as was the case in this example.

3.

$\int \sin x \cdot (\cos x)^{½}\, dx$ choice: $t = \cos x$ =>

$\frac{dt}{dx} = -\sin x$ ⇔

$dx = \frac{dt}{-\sin x}$

$\int \sin x \cdot t^{½} \cdot \frac{dt}{-\sin x}$ =

$- \int t^{½}\, dt$ =

$-\frac{2}{3} t^{\frac{3}{2}} + k_t$ =

$-\frac{2}{3} (\cos x)^{\frac{3}{2}} + k_x$ which is the answer

Integration by parts

Integration by parts can be used, when x is in two multiplied parts (u and v) of the whole function (f):

$\int u \cdot v\,dx = U \cdot v - \int U \cdot v'\,dx$ U is the base function of u

The formula is proved by differentiating the right side:

$(U \cdot v - \int U \cdot v'\,dx)' = (U \cdot v)' - (\int U \cdot v'\,dx)' =$ part by part

$(U' \cdot v + U \cdot v') - (U \cdot v') =$ product rule and ´ repeals ∫

$U' \cdot v = u \cdot v$

As by integration gives the left side. Thus proven.

Examples

1.

$\int x \cdot \sin x\,dx = (-\cos x) \cdot x - \int(-\cos x) \cdot 1\,dx = -x \cdot \cos x + \sin x + k$

2.

And now an advanced solution. We want to calculate

$\int e^x \cdot \sin x\,dx$ and do so by integration by parts:

$\int e^x \cdot \sin x\,dx = e^x \cdot \sin x - \int e^x \cdot \cos x\,dx$ *equation 1*

Here we get nowhere, but if we use partial integration once more on the latter part:

$\int e^x \cdot \cos x\,dx = e^x \cdot \cos x - \int e^x \cdot (-\sin x)\,dx$

and insert it into *equation 1*:

$\int e^x \cdot \sin x\,dx = e^x \cdot \sin x - (e^x \cdot \cos x - \int e^x \cdot (-\sin x)\,dx) \Leftrightarrow$

and collect all the integrals on the left side:

$$\int e^x \cdot \sin x\, dx + \int e^x \cdot \sin x\, dx = e^x \cdot \sin x - e^x \cdot \cos x \quad \Leftrightarrow$$

$$2 \int e^x \cdot \sin x\, dx = e^x \cdot \sin x - e^x \cdot \cos x + k \quad \Leftrightarrow$$

$$\int e^x \cdot \sin x\, dx = \frac{1}{2}(e^x \cdot \sin x - e^x \cdot \cos x) + k =$$

$$\frac{1}{2} e^x (\sin x - \cos x) + k$$ which is the solution.

Other examples

3.

In the chapter on differentiation, we saw an example with formulas for the free fall that has a constant acceleration, g.

Now we will consider all linear motions with a constant acceleration and the corresponding formulas of acceleration, velocity, and position as functions of time.

We start with the definitions

the momentary velocity: $v = \frac{\text{distance}}{\text{time}} = \frac{ds}{dt} \Leftrightarrow ds = v \cdot dt$

and the momentary acceleration: $a = \frac{\text{velocity}}{\text{time}} = \frac{dv}{dt} \Leftrightarrow dv = a \cdot dt$

From these expressions, we may integrate going from acceleration to velocity and on to position like this:

Acceleration $\quad a = \text{constant} \quad =>$

Velocity $\quad dv = a \cdot dt \Leftrightarrow \int dv = \int a \cdot dt \quad \Leftrightarrow$

$$v = a \int dt \Leftrightarrow v = at + k \quad \Leftrightarrow$$

$$v = at + v_0$$

Position $\quad ds = v \cdot dt \quad \Leftrightarrow \quad \int ds = \int v \cdot dt \quad \Leftrightarrow$

$$s = \int (at + v_0)\, dt \quad \Leftrightarrow$$

$$s = \frac{1}{2} \cdot a \cdot t^2 + v_0 \cdot t + s_0$$

The integration constant for velocity is the initial velocity v_0

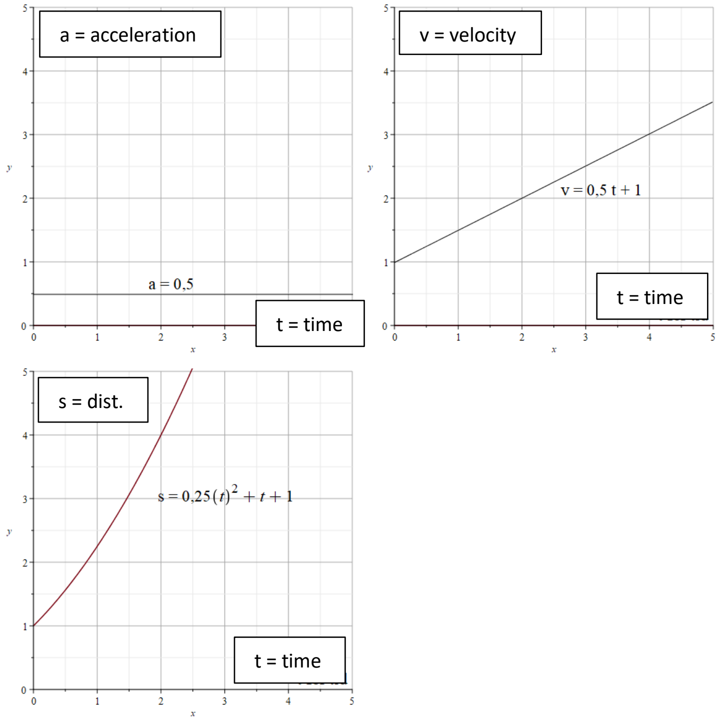
The integration constant for position is the initial position s_0

The a formula renders a horizontal line in a t,a diagram.

The v formula renders a straight line with slope a and starts at v_0 in a t,v diagram.

The s formula renders a second degree polynomial with the slope v and starts at s_0 in a t,s diagram.

See also these diagrams of the functions with numbers inserted:
$a = 0{,}5 \quad v_0 = 1 \quad s_0 = 1$



The specific integral

So far, we have considered a differentiated function return to the base function. We have gathered the very, very little pieces into the whole. The technical term for this is *the indeterminate integral*, which gives us the complete solution.

Maybe we are only interested in a part of the whole. For instance we may be ignorant of the past, and only focus on the future. Then we use *the specific integral* which yields a specific/particular solution.

We write it, like this

$$y = \int_a^b f(x) \cdot dx$$

Here it is stated that we find y by gathering the very small parts $f(x) \cdot dx$ from $x = a$ (the lower limit) to $x = b$ (the upper limit).

When we have integrated and found the base function F(x), we insert b for x and subtract a inserted for x. We write it, this way

$$y = \int_a^b f(x) \cdot dx = [F(x)]^b{}_a = F(b) - F(a) = \text{answer}$$

for instance

$$y = \int_1^3 2x \cdot dx = [x^2]^3{}_1 = 3^2 - 1^2 = 8$$

Here we have integrated the function 2x going from 1 to 3.

The calculation rules are the same as for the indeterminate integral. Yet, we must note two things:

First, the integration constant k comes both at F(b) and F(a), but since we have upper limit minus lower limit, we also have k minus k, which is zero. Thus, no k.

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Second, if we integrate by substitution, the limits also substitutes. We will do so in an example.

Examples

1.

$\int_1^2 (\ln x + \ln x^2)\,\mathrm{dx} \;=\; \int_1^2 (\ln x + 2\ln x)\,\mathrm{dx} \;=$ for $x > 0$

$\int_1^2 (3\ln x)\,\mathrm{dx} \;=\; 3\int_1^2 \ln x\,\mathrm{dx} \;=\; 3\cdot[x\ln|x| - x]^2{}_1 \;=$

upper minus lower

$3((2\ln 2 - 2) - (1\ln 1 - 1)) \;=\; (6\ln 2 - 6) - (-3) \;\approx\; 1.16$

2.

$\int_{-1}^0 \frac{2x}{x^2-3}\,\mathrm{dx}$ and $x \neq \sqrt{3}$ substitution, choice: $t \;=\; x^2 - 3 \;\Rightarrow$

$\frac{\mathrm{dt}}{\mathrm{dx}} \;=\; 2x \;\Leftrightarrow$

$\mathrm{dx} \;=\; \frac{\mathrm{dt}}{2x}$

changing limits: $t_{lower} \;=\; (-1)^2 - 3 \;=\; -2$

$t_{upper} \;=\; 0^2 - 3 \;=\; -3$

substitution of dx and limits:

$\int_{-2}^{-3} \frac{2x}{t} \cdot \frac{\mathrm{dt}}{2x} \;=$

$\int_{-2}^{-3} \frac{1}{t} \cdot \mathrm{dt} \;=$ and we have a full t expression

$[\ln|t|]^{-3}{}_{-2} \;=$

$\ln |-3| - \ln |-2| =$

$\ln \frac{3}{2} \approx 0.406$ which is the answer

We do not need to substitute back to a x expression, since we also substituted the limits and inserted the figures.

Now we solve the same problem by substituting back again to a x expression:

$\int_{-1}^{0} \frac{2x}{x^2-3} dx$ choice: $t = x^2 - 3 \Rightarrow$

$\frac{dt}{dx} = 2x \Leftrightarrow$

$dx = \frac{dt}{2x}$

Now we will substitute from x to t and should substitute the limits from x to t, too, but since we will go back later, we just call the t limits some unknown values, like a and b, meanwhile:

and we insert

$\int_{a}^{b} \frac{2x}{t} \cdot \frac{dt}{2x} =$

$\int_{a}^{b} \frac{1}{t} \cdot dt =$ a t expression

$[\ln |t|]^{b}_{a} =$

$[\ln |x^2 - 3|]^{0}_{-1} =$ back to x with x limits

$(\ln |0^2 - 3|) - (\ln |(-1)^2 - 3|) =$

$\ln 3 - \ln 2 =$

$\ln \frac{3}{2} \approx 0.406$ same answer

3.

A car starts accelerating

$a(t) = \frac{\sqrt{t}}{10}$ where t is time in seconds

What is the velocity v after 60 seconds, and how far is the car, distance s, in meters?

We integrate from acceleration to velocity, and on to distance:

$a = \frac{dv}{dt}$ => $dv = a\,dt$ => a-function inserted

$v = \int_0^{60} \frac{\sqrt{t}}{10}\,dt = \frac{1}{10}\int_0^{60} t^{½}\,dt = \frac{1}{10} \cdot \left(\frac{2}{3}\left[t^{\frac{3}{2}}\right]^{60}_{0} \right) =$

$\frac{1}{10} \cdot \left(\frac{2}{3}(60^{\frac{3}{2}})\right) - (0) = 31\,\frac{m}{s}$ $\left(\approx 110\,\frac{km}{hour}\right)$ which is the velocity

$v = \frac{ds}{dt}$ => $ds = v\,dt$ => v-function inserted

$s = \int_0^{60}\left[\frac{1}{10} \cdot \frac{2}{3}(t^{\frac{3}{2}})\right] dt = \frac{1}{10} \cdot \frac{2}{3}\int_0^{60} t^{\frac{3}{2}}\,dt = \frac{1}{10} \cdot \left(\frac{2}{3} \cdot \frac{2}{5} \cdot \left[t^{\frac{5}{2}}\right]^{60}_{0} \right)$
$=$

$\frac{1}{10} \cdot \left(\frac{2}{3} \cdot \frac{2}{5}(60^{\frac{5}{2}})\right) - (0) \approx 744\text{ m}$ which is the distance

This example will be further considered in the chapter ”Areas” *example 4.*

Areas

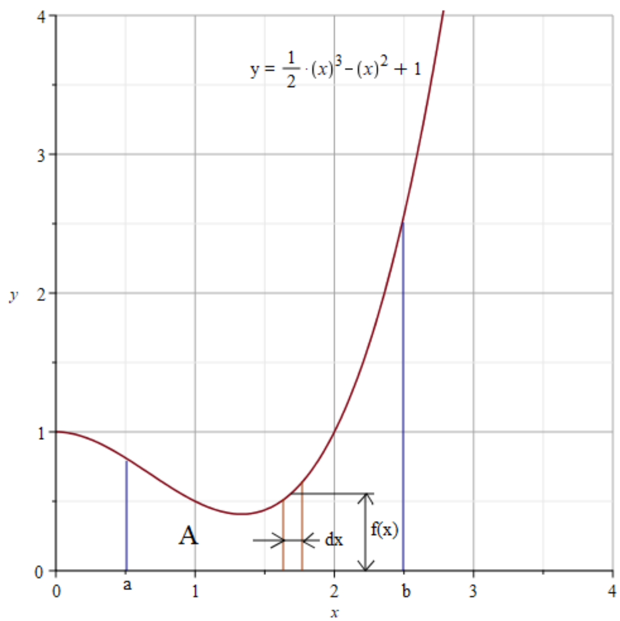
Often in mathematics, after a formula has been derived, and a tool has been developed, it turns out that the tool may be used for something different.

The specific integral can also be used as an advanced method to find areas of otherwise ”impossible” figures.

Let us consider this expression again - yet in another way:

$A \;=\; \int_a^b f(x) \cdot dx$ now called A

dx is a very small distance in the x direction, while f(x) is the corresponding distance in the f(x) direction (y direction). Multiplication: $f(x) \cdot dx$ form a very small area. If we gather all the micro areas (very small strips) from a to b, we have a visible macro area. The height f(x) of the strips vary with the function, see the following example in the diagram



Since dx is infinitely small, f(x) in practice will be the height of the strip in the middle as well as in the two sides. Only, here dx is shown wide enough for us to see it.

The area limited by the x-axis, the curve, and by the lines x = a, and x = b, can be calculated precisely using the specific integral.

Examples

1.

The area in the diagram is

$$A = \int_a^b f(x) \cdot dx \qquad \Leftrightarrow$$

$$A = \int_{0,5}^{2,5} \left(\frac{1}{2}x^3 - x^2 + 1\right) dx \qquad \Leftrightarrow$$

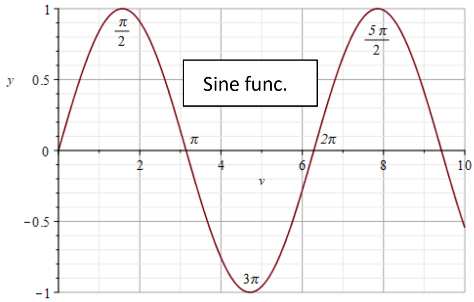
$$A = \left[\frac{1}{2} \cdot \frac{1}{4} \cdot x^4 - \frac{1}{3}x^3 + x\right]_{0,5}^{2,5} \qquad \Leftrightarrow$$

$$A \approx (4,88 - 5,21 + 2,5) - (0,0078 - 0,0417 + 0,5) \qquad \Leftrightarrow$$

$$A \approx (2,17) - (0,466) \approx 1,704$$

2.

If we are below the x-axis, the function value f(x) is negative, and the area will become negative as well. We therefor do numerical calculation if the area is below the x-axis. For instance, if we will find the area between the x-axis and the sine curve from x = 0 to x = 2π



$$A = \int_0^{\pi} \sin x \cdot dx + \left| \int_{\pi}^{2\pi} \sin x \cdot dx \right| \qquad \Leftrightarrow$$

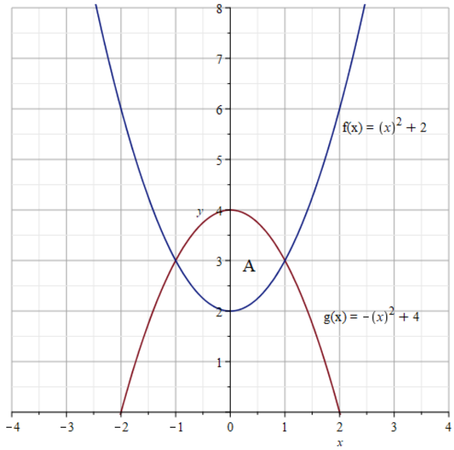
$$A = (-\cos \pi - (-\cos 0)) + |(-\cos 2\pi - (-\cos \pi))| \qquad \Leftrightarrow$$

$$A = (-(-1) - (-1)) + |(-1 - (-(-1))| = 2 + 2 \qquad \Leftrightarrow$$

$$A = 4$$

3.

Let us find the common area between these two parabolas $f(x) = x^2 + 2$ and $g(x) = -x^2 + 4$



We find the limits, where the parabolas intersect, i.e.

$f(x) = g(x)$ =>

$x^2 + 2 = -x^2 + 4 \quad \Leftrightarrow$

$2x^2 - 2 = 0 \quad \Leftrightarrow$

$x = -1$ and $x = 1$

Now we can integrate by finding the area under g and subtract the area under f

$$A = \int_{-1}^{1} (-x^2 + 4)\,\mathrm{dx} - \int_{-1}^{1} (x^2 + 2)\,\mathrm{dx} \quad \Leftrightarrow$$

$$A = \left[(-\tfrac{1}{3}x^3 + 4x) - (\tfrac{1}{3}x^3 + 2x)\right]^{1}_{-1} \quad \Leftrightarrow$$

$$A = ((-\tfrac{1}{3} + 4) - (\tfrac{1}{3} + 2)) - ((\tfrac{1}{3} - 4) - (-\tfrac{1}{3} - 2)) \quad \Leftrightarrow$$

$$A = \tfrac{4}{3} - (-\tfrac{4}{3}) \quad \Leftrightarrow$$

$$A = \tfrac{8}{3}$$

4.

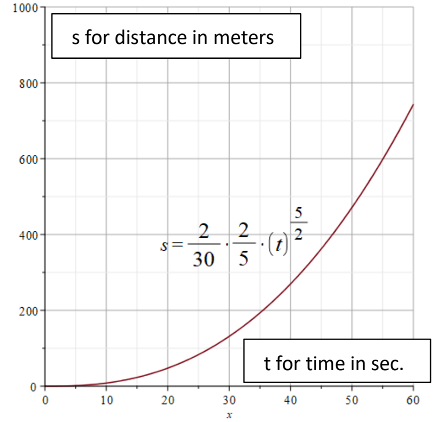
We will continue *example 3* from chapter: ”The specific integral”, with an accelerating car:

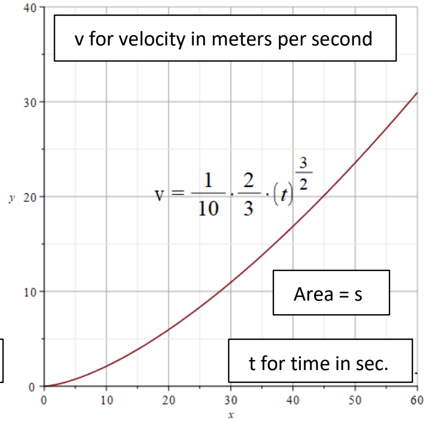
We had: $a = \frac{\sqrt{t}}{10} \quad => \quad v = \frac{1}{10} \cdot \frac{2}{3} \cdot t^{\frac{3}{2}} \quad => \quad s = \frac{1}{10} \cdot \frac{2}{3} \cdot \frac{2}{5} \cdot t^{\frac{5}{2}}$

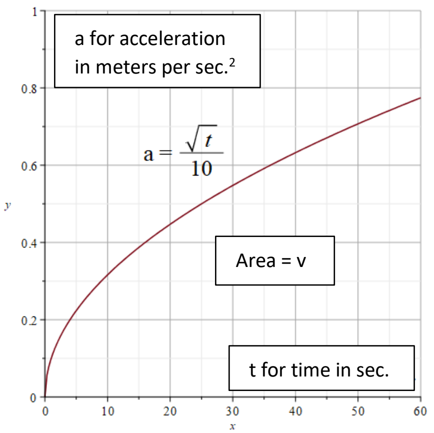
Now we are able to find the velocity v graphically/numerically by reading the area under the t,a curve. From t = 0 to t = 60 seconds, the area observes to correspond to 31 meters per second.

And we may find the distance s by reading the area under the t,v curve. From t = 0 to t = 60 seconds, the area observes to correspond to 744 meters.

Clearly, there are uncertainties in our readings, yet we observe correspondence.



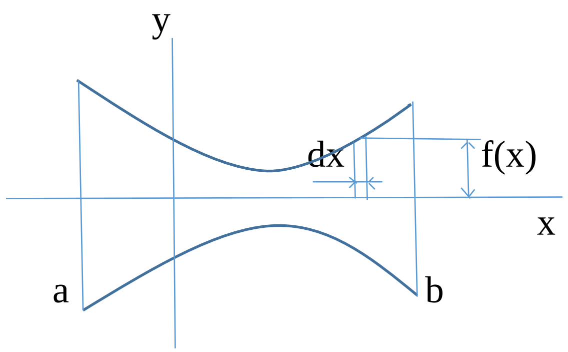




Volumes

We can rotate a 2D area around the x or y-axis and have a 3D volume.

The formula for rotation around the x-axis derives



If we rotate our infinitesimally thin strip around the x-axis we have a micro cylinder. A macro cylinder has the volume

$V = \pi \cdot r^2 \cdot l$ l for length

for our micro cylinder the volume is

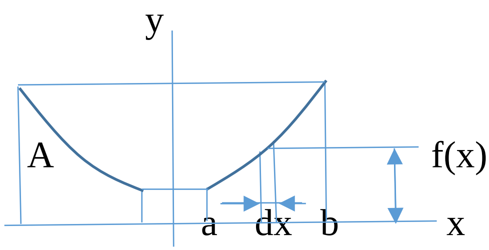
$dV = \pi \cdot f(x)^2 \cdot dx$

by integration (gathering all micro cylinders) from a to b

$V = \pi \cdot \int_a^b f(x)^2 \, dx$ the rotation volume around the x-axis

Thus, the volume can be calculated when we have an expression of the function, which informs how the radius varies.

The formula for rotation around the y-axis derives



By rotation of our infinitely thin strip around the y-axis, we have a cylinder shell with volume

dV = height · circumference · micro-thickness =>

$dV = f(x) \cdot 2\pi x \cdot dx$ =>

and when we integrate (gather all the micro cylinder shells) from a to b, the volume - calculated numerically (x or f(x) may be negative) - is

$V = \left| 2\pi \cdot \int_a^b x \cdot f(x)\, dx \right|$ the rotation volume around the y-axis

For the figure shown the rotation volume looks like the space under the stands in a stadium.

We may also view the rotation volume as the area A rotated around the y-axis.

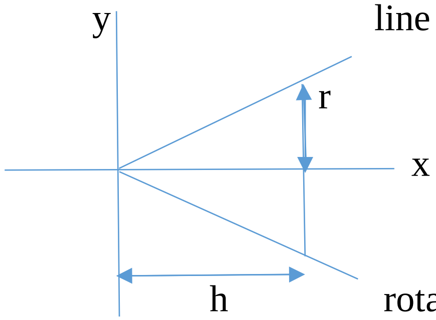
If $a = 0$ there will be no hole in the middle.

Examples

1.

We will find the formula of a cones volume.

We rotate a line segment once around the x-axis, and have a cone that lies down.



line segment: $f(x) = a \cdot x = \frac{r}{h} \cdot x$

rotated line segment

$$V = \pi \cdot \int_a^b f(x)^2 \, dx \qquad =>$$

$$V = \pi \cdot \int_0^h \left(\frac{r}{h} \cdot x\right)^2 dx \qquad \text{r and h are constants} \qquad \Leftrightarrow$$

$$V = \pi \cdot \left(\frac{r}{h}\right)^2 \cdot \int_0^h x^2 \, dx \qquad \Leftrightarrow$$

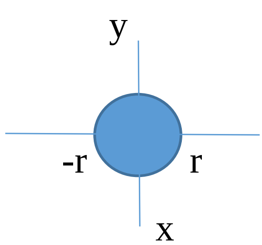
$$V = \pi \cdot \left(\frac{r}{h}\right)^2 \cdot \left[\frac{x^3}{3}\right]_0^h \qquad \Leftrightarrow$$

$$V = \pi \cdot \left(\frac{r}{h}\right)^2 \cdot \left(\frac{h^3}{3} - 0\right) \qquad \Leftrightarrow$$

$$V = \frac{\pi}{3} r^2 h$$ which is the formula for a cones volume

2.

Also, let us find the volume of a sphere:



A circle at the centre of the coordinate system has the equation $x^2 + y^2 = r^2$

A whole circle can only be described as a parameter function (see the chapter about vector functions). As an ”ordinary” function we have to make the equation for a half circle above the x-axis

$r^2 = x^2 + y^2 \quad \Leftrightarrow \quad y = (r^2 - x^2)^{½} \quad \Leftrightarrow \quad f(x) = (r^2 - x^2)^{½} \quad =>$

This half circle is rotated once around the x-axis, while the limits are -r and r

$$V = \pi \cdot \int_{-r}^{r} f(x)^2 \, dx \qquad =>$$

$$V = \pi \cdot \int_{-r}^{r} ((r^2 - x^2)^{½})^2 \, dx \qquad \Leftrightarrow$$

$$V = \pi \cdot \int_{-r}^{r} (r^2 - x^2) \, dx \qquad \text{split in two} \qquad \Leftrightarrow$$

$$V = \pi \cdot \int_{-r}^{r} r^2 \, dx \; - \; \pi \cdot \int_{-r}^{r} x^2 \, dx \qquad r^2 \text{ is a constant} \qquad \Leftrightarrow$$

$$V = (\pi \cdot r^2 \, [x]^{r}_{-r}) - (\pi \cdot \left[\frac{1}{3} x^3\right]^{r}_{-r}) \qquad \Leftrightarrow$$

$$V = (\pi \cdot r^3 - (-\pi \cdot r^3)) - (\pi \cdot (\frac{1}{3} r^3 - (-\frac{1}{3} r^3))) \qquad \Leftrightarrow$$

$$V = 2\pi \cdot r^3 - \frac{2}{3} \cdot \pi \cdot r^3 \qquad \Leftrightarrow$$

$$V = \frac{4}{3} \cdot \pi \cdot r^3 \qquad \text{which is the volume of a sphere}$$

3.

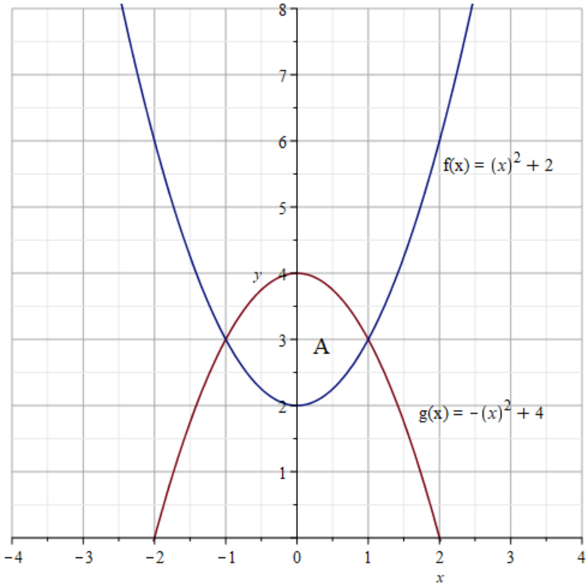
We may also find the rotation volume between two curves. Here we will do it for the two parabolas in the recent example 3 of the former chapter. We rotate around the y-axis:

The function f(x) here becomes ”upper minus lower”:

$(-x^2 + 4) - (x^2 + 2) \qquad =>$

and the limits are from 0 to 1, - since it is half the common area that is to be rotated once.

$$V = 2\pi \cdot \int_a^b x \cdot f(x)\, dx \qquad =>$$



$$V = 2\pi \cdot \int_{-0}^{1} x \cdot ((-x^2 + 4) - (x^2 + 2))\, dx \qquad \Leftrightarrow$$

$$V = 2\pi \cdot \int_0^1 (-2x^3 + 2x)\, dx \qquad \Leftrightarrow$$

$$V = 2\pi \left[-\frac{1}{2}x^4 + x^2\right]_0^1 \qquad \Leftrightarrow$$

$$V = 2\pi \left(\left(-\frac{1}{2} + 1\right) - (0)\right) \qquad \Leftrightarrow$$

$$V = 2\pi \cdot \frac{1}{2} \qquad \Leftrightarrow$$

$V = \pi \approx 3.14$ which is the volume

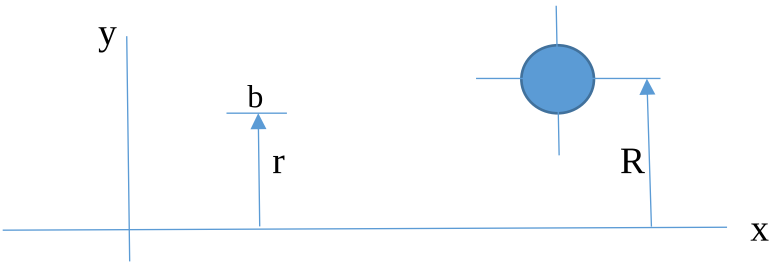
Guldin's rules

Guldin´s rules are also based on rotating a figure around an axis, which it is not intersecting. There are two rules:

1. Rotation of a curve segment, which will render an area.
2. Rotation of an area, which will render a volume.

Examples

1.



A line segment is shown to the left. The width is b. Rotation around the x-axis renders a flat belt. The area of the belt is

$A \ = \ b \cdot \text{circumference}$ $\qquad\qquad$ =>

$A \ = \ b \cdot 2\pi r$

Which is Guldin's first rule. It is also valid for curve segments where r is the radius of rotation for the centre of gravity of the curve. Yet, determination of centres of gravity is not a subject of this book.

If we will produce a flat belt where $b = 20$ mm and $r_{middle} = 300$ mm, the average (along the neutral line) area of the belt becomes:

$A_{average} \ = \ 20 \cdot 2\pi \cdot 300 \ \approx \ 37\ 700$ mm

----

A circle with radius r is shown to the right, which becomes a ring named a toroid (a ”donut”) on rotation around the x-axis. The volume of the toroid becomes

$V \ = \ A \cdot \text{circumference} \qquad\qquad =>$

$V \ = \ A_{\text{circle}} \cdot 2\pi R \qquad\qquad =>$

$V \ = \ \pi r^2 \cdot 2\pi R$

Which is Guldin’s second rule. It is also valid for unsymmetrical areas, where A calculates accordingly, and where R is the rotation radius of the centre of gravity of the area. Yet, determination of centres of gravity is not a subject of this book.

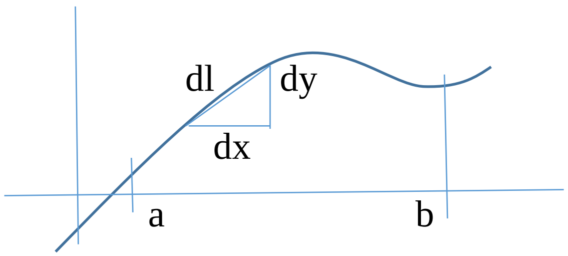
If we will produce rubber o-rings with a radius of $r = 3$ mm and a Radius of the centre line of $R = 25$ mm the volume of the o-ring is

$V \ = \ (\pi \cdot 3^2) \cdot (2\pi \cdot 25)$

$V \ \approx \ 4\,441 \text{ mm}^3$

For instance, this information may be used to calculate how much raw rubber powder is needed for the production.

Curve length



We use Pythagoras on the infinitesimal small rectangular triangle, where dl is a secant of the curve:

$(dl)^2 = (dx)^2 + (dy)^2$ and $dy = f'(x) \cdot dx$

which is inserted =>

$(dl)^2 = (dx)^2 + (f'(x) \cdot dx)^2$ ⇔

$dl = \sqrt{(dx)^2 + (f'(x) \cdot dx)^2)}$ ⇔

$dl = \sqrt{(1 + f'(x)^2) \cdot (dx)^2}$ ⇔

$dl = \sqrt{1 + f'(x)^2} \cdot dx$ =>

$l = \int_a^b \sqrt{1 + f'(x)^2}\, dx$ or

$l = \int_a^b (1 + f'(x)^2)^{½}\, dx$ which is the curve length from a to b

So, here, just like for areas and volumes, we can calculate precisely, provided the curve/figure is written as a function.

Example

We will find the curve length of the function $f(x) = \frac{1}{3}x^{\frac{3}{2}}$ from $x = 0$ to $x = 2$:

$l = \int_a^b (1 + f'(x)^2)^{½}\,dx$ where

$f'(x) = \frac{1}{2} \cdot x^{\frac{1}{2}}$ =>

$\int_0^2 (1 + (\frac{1}{2} \cdot x^{\frac{1}{2}})^2)^{½}\,dx$ ⇔

$\int_0^2 (1 + \frac{x}{4})^{½}\,dx$ choice: $t = 1 + \frac{x}{4}$

$\frac{dt}{dx} = \frac{1}{4}$

$dx = 4 \cdot dt$

$4\int_a^b t^{½}\,dt = 4\left[\frac{2}{3} \cdot t^{\frac{3}{2}}\right]_a^b = 4\left[\frac{2}{3}(1 + \frac{x}{4})^{\frac{3}{2}}\right]_0^2 = 4\,[(\frac{2}{3} \cdot (\frac{3}{2})^{\frac{3}{2}}) - (\frac{2}{3})]$

≈ 2.24 which is the curve length

Often it is very difficult to calculate the curve lengths, so usually CAS is applied.

Differential equations

An equation with both a quantity (y) and its differential coefficient (y´) is called a differential equation. It describes a quantity, and how it changes, - usually related to time. We now talk about advanced mathematics to be used in complicated problems.

Some technical terms:

A differential equation is solved by integration calculus. Thus, just like in other integrals we have an *indeterminate solution = complete solution = general solution* with an unknown integration constant - or a *special solution = particular solution* where the integration constant is shortened out.

A differential equation with *y´* (first derivative) is named a *first order differential equation*. A differential equation with *y´´* (second derivative) is named *a second order differential equation.*

A differential equation with just y-parts (i.e. with y, y´, y´´…) is called *homogeneous* - otherwise it is *inhomogeneous*.

Typical differential equations

We will derive and prove the formulas for solving one basic, and four typical differential equations. The fourth formula solves many differential equations. Furthermore, we will derive and prove the formula for solving a special type called *the logistic differential equation*.

The basic differential equation is

$$\frac{dy}{dx} = k \cdot y \qquad\qquad =>$$

which is solved by separating the variables

$$\frac{dy}{y} = k \cdot dx$$

Here we call it *theorem 0*, and we use it in example 0:

Example 0

How can we predict the decay of radioactive matter with time?

We do so by observing small changes taking place presently and then integrate for the full picture in past and future. Surely, there is uncertainty involved, but here we present the basics:

The activity A of a radioactive matter equals a decay constant k, times the number of radioactive atoms, N, in a sample

$A = k \cdot N$

The activity also equals the change of the number of radioactive atoms, dN, in infinitesimal time, dt

$A = -\frac{dN}{dt}$ minus because the activity decreases =>

the right sides are alike

$-\frac{dN}{dt} = k \cdot N$ ⇔

here we separate the variables, i.e. we gather N to the left and t to the right

$\frac{dN}{N} = -k \cdot dt$ ⇔

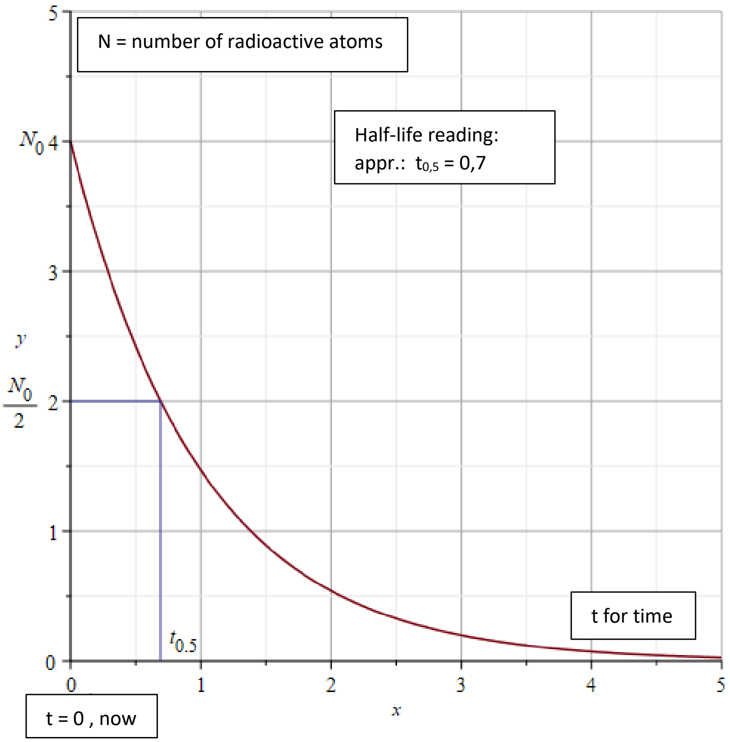
and if N_0 is the number of radioactive atoms at time 0 (now) and t is time to come, we have

$\int_{No}^{N} \frac{1}{N}\, dN = -k \int_{o}^{t} dt$ ⇔ $\ln N - \ln N_0 = -k \cdot t$ ⇔

$\ln \frac{N}{No} = -k \cdot t$ ⇔ $\frac{N}{No} = e^{-k \cdot t}$ ⇔

$N = N_0 \cdot e^{-k \cdot t}$ which is the answer.

In a diagram, the curve will look this way in principle. The curve becomes quantified when/if we know k.



The curve is exponentially decreasing and asymptotic to the first-axis. The radioactivity never becomes zero. Therefore, it is nice to know when we have reached half the number of radioactive atoms $\frac{No}{2}$. The corresponding amount of time is named the half-life, as shown. This way, we may compare the half-life for various radioactive materials. For some radioactive materials like platinum-178, the decay is quick and is measured in seconds, while others, for instance certain types of uranium, decay over millions of years.

More theory

The four typical differential equations are:

(Please note that some multiplication signs (dots) are omitted)

Equation		Solution formula
$y' + ay = 0$	=>	$y = c \cdot e^{-ax}$
$y' + ay = b$	=>	$y = \frac{b}{a} \cdot c \cdot e^{-ax}$
$y' + ay = h(x)$	=>	$y = e^{-ax} \int h(x) \cdot e^{ax}\, dx + c \cdot e^{-ax}$
$y' + g(x) \cdot y = h(x)$	=>	$y = e^{-G(x)} \int h(x) \cdot e^{G(x)}\, dx + c \cdot e^{-G(x)}$

Theorem 3 and 4 hold functions that may be comprehensive on their own. Thus, we are able to do calculations on very complicated systems/models like economic models, climate models, etc.

It is easier to present the solution formulas in reverse order.

Theorem 4

$y' + g(x) \cdot y = h(x)$ and multiplied by $e^{G(x)}$ on either side =>

$$y' \cdot e^{G(x)} + g(x) \cdot y \cdot e^{G(x)} = h(x) \cdot e^{G(x)}$$

here we utilize a known formula for differentiation of a product:

$$(y \cdot e^{G(x)})' = y' \cdot e^{G(x)} + y \cdot g(x) \cdot e^{G(x)}$$

Where the right side equals the left side above. Thus, the right side above must also equal the left side below. We continue with the latter:

$(y \cdot e^{G(x)})' \ = \ h(x) \cdot e^{G(x)}$

and integrate on either side (remembering the integration constant c)

$(y \cdot e^{G(x)})' \ = \ h(x) \cdot e^{G(x)} \qquad \Leftrightarrow$

$y \cdot e^{G(x)} \ = \ \int h(x) \cdot e^{G(x)} \, dx + c \qquad \Leftrightarrow$

$y \ = \ e^{-G(x)} \int h(x) \cdot e^{G(x)} \, dx + c \cdot e^{-G(x)}$ *theorem 4*

Theorem 3

$y' + ay \ = \ h(x)$

Now g(x) is a constant a, Thus, $G(x) = ax$ which is inserted directly into theorem 4

$y \ = \ e^{-ax} \int h(x) \cdot e^{ax} \, dx + c \cdot e^{-ax}$ *theorem 3*

Theorem 2

$y' + ay \ = \ b$

Now h(x) also equals a constant b, which is inserted directly into theorem 3

$y \ = \ e^{-ax} \int b \cdot e^{ax} \, dx + c \cdot e^{-ax} \qquad \Leftrightarrow$

$y \ = \ e^{-ax} \cdot b \cdot \frac{1}{a} \cdot e^{ax} + c \cdot e^{-ax}$

$y \ = \ \frac{b}{a} + c \cdot e^{-ax}$ *theorem 2*

Theorem 1

$y' + ay \ = \ 0$

b = 0 inserted in theorem 2

$y = c \cdot e^{-ax}$ *theorem 1*

In some tables ay is moved to the right side and -a is called k, which renders:

$y' = ky \quad \Rightarrow \quad y = c \cdot e^{kx}$ *theorem 1*

Note, that written this way, the differential equation in *theorem 1* equals the basic differential equation *theorem 0*, which was solved by separating the variables (and which was used in example 0). Thus, there are two solution methods:

1.

We will now solve the differential equation from example 0 by using theorem 1

$$-\frac{dN}{dt} = k \cdot N \quad \Leftrightarrow \quad N' = -k \cdot N \quad \Rightarrow$$

$$N = c \cdot e^{-kt}$$

which is the solution for the indeterminate integral.

In example 1 we had

$$N = N_0 \cdot e^{-kt}$$

by solving a specific integral.

The difference is that we know the initial value of N, N_0 , which we use in the specific integral. This information we did not have for the indeterminate integral, so here we could only continue with new information. However, we find that the solutions are the same in principle.

Also, here we get the explanation to the fact that the indeterminate integral yields a complete solution, while the specific integral yields a specific solution.

We continue with the indeterminate solution by inserting the initial value of N, N_0 , at t = 0

$t = 0 \;\Rightarrow\; N = N_0$ inserted =>

$N_0 = c \cdot e^{-k0} = c \cdot 1$ => $c = N_0$ =>

$N = N_0 \cdot e^{-kt}$ same answer

2.

A big cup of coffee with the temperature 83°C, which is in a room with a constant temperature of 22°C, follows the differential equation

$$\frac{dT}{dt} = -k\cdot(T - 22)$$

where T is temperature in °C, t is time in minutes and k is a constant.

It was measured, that the coffee is 65° after 20 minutes.

What is the equation for T as a function of time?

When is the coffee 45° ?

We rearrange the equation

$$\frac{dT}{dt} = -k\cdot(T-22) \qquad \Leftrightarrow$$

$$\frac{dT}{dt} + k\cdot T = k\cdot 22$$

and find that it corresponds with theorem 2

$y' + ay = b \quad \Rightarrow \quad y = \frac{b}{a} + c \cdot e^{-ax}$

which, in our case becomes

$\frac{dT}{dt} + k \cdot T = k \cdot 22 \quad \Rightarrow \quad T = \frac{k \cdot 22}{k} + c \cdot e^{-kt} = 22 + c \cdot e^{-kt}$

We find c from the information: T = 83 when t = 0 =>

$T = 22 + c \cdot e^{-kt} \quad \Rightarrow \quad 83 = 22 + c \cdot e^{0} \quad \Rightarrow$

$c = 61 \quad \Rightarrow \quad T = 22 + 61 \cdot e^{-kt}$

We find k from the information: T = 65 when t = 20

$T = 22 + 61 \cdot e^{-kt} \quad \Rightarrow \quad 65 = 22 + 61 \cdot e^{-k \cdot 20}$

$\Leftrightarrow \quad \frac{65-22}{61} = e^{-20k}$

$\Leftrightarrow \quad \ln 0.7049 = -20k$

$\Leftrightarrow \quad k = \frac{-0.3497}{-20} = 0.0175 \Rightarrow$

$T = 22 + 61 \cdot e^{-0.0175 \cdot t}$ which is the cooling function.

and for T = 45°

$45 = 22 + 61 \cdot e^{-0.0175 \cdot t} \quad \Leftrightarrow \quad \ln \frac{45-22}{61} = -0.0175 \cdot t \quad \Leftrightarrow$

$t = \frac{-0.9754}{-0.0175} = 55.7$ minutes which is the answer

3.

In a brewery is produced mineral water. In a pressure vessel CO_2 is dissolved in the water, as described in this differential equation

$\frac{dC}{dt} = k \cdot (C_s - C)$

Where $\frac{dC}{dt}$ is the growth in concentration per unit time, k is a constant, C_s is the saturation concentration and C is the variable concentration.

We will find an expression for the concentration C as a function of time (i.e. the indeterminate solution) by solving the differential equation.

We rearrange so we can correspond with the formulas

$$\frac{dC}{dt} = k \cdot (C_s - C) \quad \Leftrightarrow \quad \frac{dC}{dt} = k \cdot C_s - k \cdot C \quad \Leftrightarrow$$

$$\frac{dC}{dt} + k \cdot C = k \cdot C_s$$

We find correspondence with theorem 2

$$y' + ay = b \quad \Rightarrow \quad y = \frac{b}{a} + c \cdot e^{-ax}$$

which in our case becomes

$$\frac{dC}{dt} + k \cdot C = k \cdot C_s \quad \Rightarrow \quad C = \frac{k \cdot C_s}{k} + c \cdot e^{-kt} = C_s + c \cdot e^{-kt}$$

We find c from the information: $C = 0$ when $t = 0$ =>

$$0 = C_s + c \cdot e^{-k \cdot 0} \quad \Leftrightarrow \quad c = -C_s \quad \Rightarrow$$

$$C = C_s - C_s \cdot e^{-kt}$$

Which is the equation for the concentration growth.

4.

Now we will solve a difficult problem:

At the launch of a rocket was measured data that complied the following differential equation describing velocity as function of time (v = f(t)), valid for the first 14 seconds:

$\frac{dv}{dt} - \frac{1}{15-t} \cdot v = \frac{300}{15-t} - 9.81$ *(equation from: www.studieportalen.dk)*

v is the velocity of the rocket in meters per second, and t is time in seconds. At the start: t = 0 and v = 0

We will find an expression for the velocity v as function of time, i.e. we will solve the differential equation.

We compare with theorem 4

$y' + g(x) \cdot y = h(x)$ => and in our case:

$v' + g(t) \cdot v = h(t)$ which is compared with

$v' - \frac{1}{15-t} \cdot v = \frac{300}{15-t} - 9.81$ which corresponds when

$g(t) = -\frac{1}{15-t}$ and $h(t) = \frac{300}{15-t} - 9.81$ =>

therefore the solution formula is

$v = e^{-G(t)} \int h(t) \cdot e^{G(t)} \, dt + c \cdot e^{-G(t)}$

To continue we have to calculate G(t) which is the integral of (the base function for) g(t)

$G(t) = \int \left(-\frac{1}{15-t}\right) dt$ substitution, choice $s = 15 - t$

=> $\frac{ds}{dt} = -1$

⇔ $dt = -ds$

$G(t) = \int \left(-\frac{1}{s}\right)(-ds)$

$G(t) = \ln|s|$ $\Rightarrow$

The integration constant is not added since it was already taken into account, when theorem 4 was derived.

Substitute back

$G(t) = \ln|15 - t|$

Since we know that t max. is 14, 15 - t must be positive, so the numerical parenthesis becomes an ordinary parenthesis.

$G(t) = \ln(15 - t)$

and we insert

$$v = e^{-G(t)} \int h(t) \cdot e^{G(t)}\, dt + c \cdot e^{-G(t)} \quad \Rightarrow$$

$$v = e^{-\ln(15-t)} \int \left(\frac{300}{15-t} - 9.81\right) \cdot e^{\ln(15-t)}\, dt + c \cdot e^{-\ln(15-t)} \quad \Leftrightarrow$$

reduce

$$v = \frac{1}{15-t} \int (300 - 9.81(15 - t))\, dt + c \cdot \frac{1}{15-t} \quad \Rightarrow$$

integrate

$$v = \frac{1}{15-t} \left(300 \cdot t - 147.15 \cdot t + \frac{9.81}{2} \cdot t^2\right) + c \cdot \frac{1}{15-t}$$

$t = 0 \Rightarrow v = 0 \Rightarrow c = 0$ $\Rightarrow$

$$v = \frac{152.85 \cdot t + 4.905 \cdot t^2}{15-t}$$

which is the equation/expression for the velocity

The velocity after 14 seconds is:

$$v = \frac{152.85 \cdot 14 + 4.905 \cdot 14^2}{15-14}$$

$v \approx 3101$ meters per second   or ca. 11,165 kilometers per hour

The logistical differential equation

The logistical differential equation describes limited growth. The variable (here y) can reach a maximum value and no more. The equation is

$$\frac{dy}{dx} = ay(m - y) \qquad \text{or} \qquad y' = ay(m - y)$$

where m is the maximum value. We observe, that the growth $\frac{dy}{dx}$ is direct proportional to y and y's distance from the maximum value.

The solution is $$y = \frac{m}{1+c\cdot e^{-amx}}$$

Which is proved in a peculiar way: We guess the solution mentioned and control if it is true:

We differentiate the solution $$y = \frac{m}{1+c\cdot e^{-amx}} \quad =>$$

$$y' = \frac{0 - m(c\cdot e^{-amx}(-am))}{(1 + c\cdot e^{-amx})^2} = \frac{am^2(c\cdot e^{-amx})}{(1 + c\cdot e^{-amx})^2}$$

which we, together with the guessed solution, insert into the original differential equation

$$y' = ay(m - y) \quad =>$$

$$\frac{am^2(c\cdot e^{-amx})}{(1 + c\cdot e^{-amx})^2} = a \cdot \frac{m}{1+c\cdot e^{-amx}} \cdot \left(m - \frac{m}{1+c\cdot e^{-amx}}\right) \quad \Leftrightarrow$$

$$\frac{am^2(c\cdot e^{-amx})}{(1 + c\cdot e^{-amx})^2} = \frac{am^2}{1+c\cdot e^{-amx}} - \frac{am^2}{(1+c\cdot e^{-kmx})^2} \quad \Leftrightarrow$$

$$\frac{am^2(c\cdot e^{-amx})}{(1 + c\cdot e^{-mx})^2} = \frac{am^2 + am^2ec^{-amx} - am^2}{(1+c\cdot e^{-amx})^2} \quad \Leftrightarrow$$

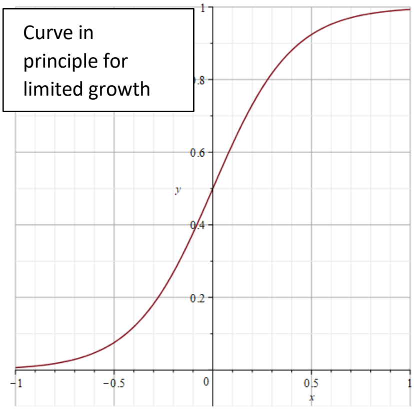
$$\frac{am^2(c \cdot e^{-amx})}{(1 + c \cdot e^{-amx})^2} = \frac{am^2 ec^{-amx}}{(1+c \cdot e^{-amx})^2} \qquad \Leftrightarrow$$

which is true.

For simplicity we chose $c = 1$, $a = 5$ and $m = 1$ in the equation

$$y = \frac{m}{1+c \cdot e^{-amx}}$$

and may sketch this curve



In the first half there is progressive growth, and in the second half there is reduced growth. The function value 1 (= 100%) is a horizontal asymptote of the function, - we never reach 1.

Example 5

Biologists have introduced 50 parrots to an island, where there were no parrots before. The biologists estimate that there may live up to 2000 parrots on the island, and after 24 months there were 100 parrots.

What is the growth function, and how long will it take until there are 1500 parrots on the island?

$y' = ay(m - y) \quad \Rightarrow \quad y = \frac{m}{1+c \cdot e^{-amt}}$

Here y = number of parrots, x is now called t for time in number of months, m is 2000, while c and k must be found from the information

$t = 0$ and $y = 50 \quad \Rightarrow \quad 50 = \frac{2000}{1+c \cdot e^{0}} \Rightarrow c = 39$

$t = 24$ and $y = 100 \quad \Rightarrow$

$100 = \frac{2000}{1+39 \cdot e^{-a \cdot 2000 \cdot 24}} \quad \Rightarrow$

$1 + 39 \cdot e^{-a \cdot 2000 \cdot 24} = 20 \quad \Leftrightarrow$

$e^{-a \cdot 2000 \cdot 24} = \frac{19}{39} \quad \Leftrightarrow$

$-a \cdot 2000 \cdot 24 = \ln \frac{19}{39} \quad \Leftrightarrow$

$a = \frac{-0.7191}{-48\,000} = 0.000014981$

Inserted we find the growth function of the parrots

$y = \frac{2000}{1+39 \cdot e^{-0.00001498 \cdot 2000 \cdot t}}$

and we expect 1500 parrots after

$1500 = \frac{2000}{1+39 \cdot e^{-0.00001498 \cdot 2000 \cdot t}} \quad \Leftrightarrow$

$39 \cdot e^{-0.00001498 \cdot 2000 \cdot t} = \frac{2000}{1500} - 1 \quad \Leftrightarrow$

$e^{-0.00001498 \cdot 2000 \cdot t} = \frac{1}{3 \cdot 39}$

$-0.00001498 \cdot 2000 \cdot t = \ln 0{,}0085 \quad \Leftrightarrow$

$t = \frac{-4.7677}{-0.03} \approx$ 158 months or appr.13 years.

Slope fields

Here is a brief description of a pretty rare concept within differential equations, namely slope fields.

Slope fields is a diagram giving a survey of possible solutions to a differential equation.

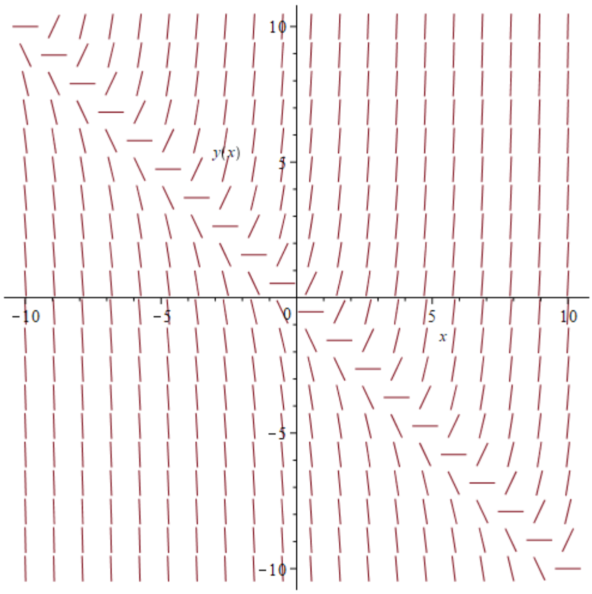
Let us consider a simple example without any function constants (such as c, k, t,…):

$y' = 2y + 2x$

Here we may insert coordinates of points (x, y) which renders the slope of the curve in that very point, for instance:

$(x, y) = (0, 0) \Rightarrow y' = 0$ or $(1, 1) \Rightarrow y' = 4$ and so on.

Then we may sketch a small tangent (= line element) in a lot of points, and consequently, we have a slope field. A tiring work suitable for CAS. Here shown for $y' = 2y + 2x$



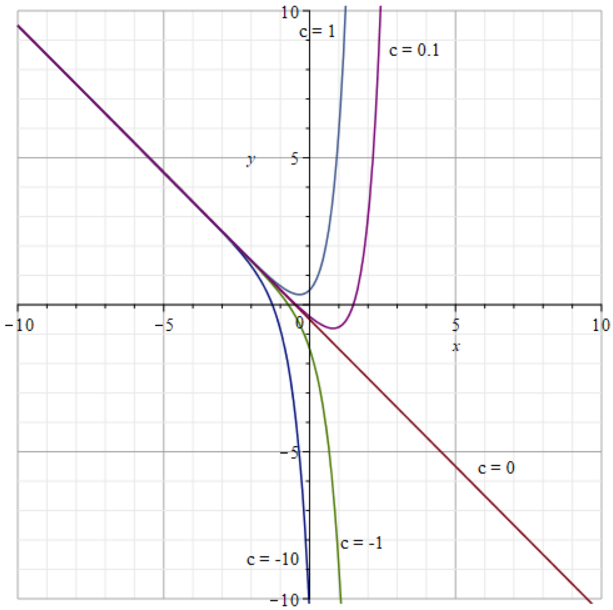
If we follow a series of small line elements, we have a certain solution curve. There exist an infinite number of solution curves. So, the slope field shows the infinite number of possible complete solutions.

We may also solve the differential equation

$y' - 2y = 2x$ which corresponds with theorem 3 and has the solution =>

$y = -x - \frac{1}{2} + c \cdot e^{2x}$ the calculation is not in focus and is not shown

and display this diagram with some c-values (various specific solutions):



The two diagrams show the same in principle:

The first diagram is based on the differential equation and displays the slope field, which gives a survey (somewhat coarse) of possible solution curves.

The other diagram is based on the complete solution and displays a few precise solution curves (specific solutions) for some values of c. One may say that we have extracted five curves from the slope field.

Usually, we solve the differential equation to find the complete solution, and then insert known values to find c, and then we have the specific solution (just like we did in the former chapter).

However, we may want a certain solution curve to pass through a certain point, for instance (0, 0). Then

- we look at the slope field and find that it seems possible
- insert (0, 0) in the solution and find c = 1
- go back and change the data (if possible) so that c becomes 1.

Seen from an overall perspective, it thus may be possible to go back and change the conditions to have a certain solution.

We also learn from the slope field how important it is to have the right rand conditions. Otherwise, we may end up having a wrong/uncertain solution.

Functions of two variables

So far we have seen functions of one variable (y depends on x or t or…). Thus, we have described most cases. However, sometimes one quantity z depends on (is a function of) two variables, x and y. Yes, there may be even more variables, but as we shall see, they are treated the same way.

Expressions of functions

If we have a function $z = f(x, y)$ and will see how z changes when only x changes, that is $\frac{dz}{dx}$, we change the notation to $\frac{\partial z}{\partial x}$
It is called the partial derivative.

This way, we state that there are other variables, but now we only focus on z related to x.

We differentiate as before regarding y (and/or other variables) as constants. All calculation rules for differentiation are the same.

Example 1

$z = 2x + 3y$ =>

$\frac{\partial z}{\partial x} = 2 + 0 = 2$ and $\frac{\partial z}{\partial y} = 0 + 3 = 3$

2.

$z = x^2 + y^3$ =>

$\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 3y^2$

This way we consider just one variable.

3D figures

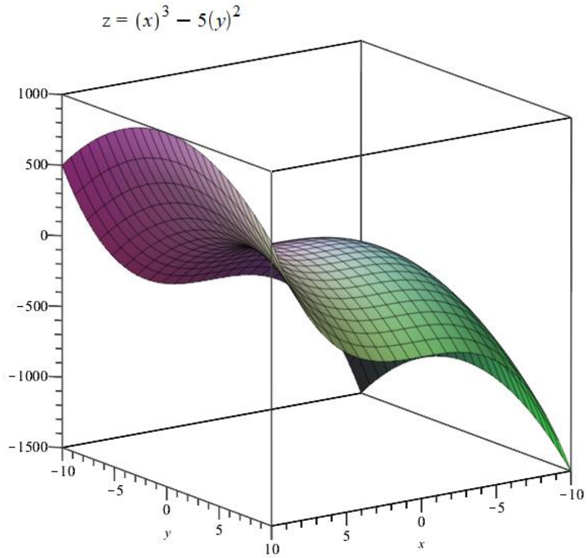
Spatial figures have equations of the type

$z = f(x, y)$

So, if we know the equation of the figure and the (x, y) coordinates of a point, we can calculate the z coordinate of the point.

Example 3.

$z = f(x,y) = x^3 - 5y^2$

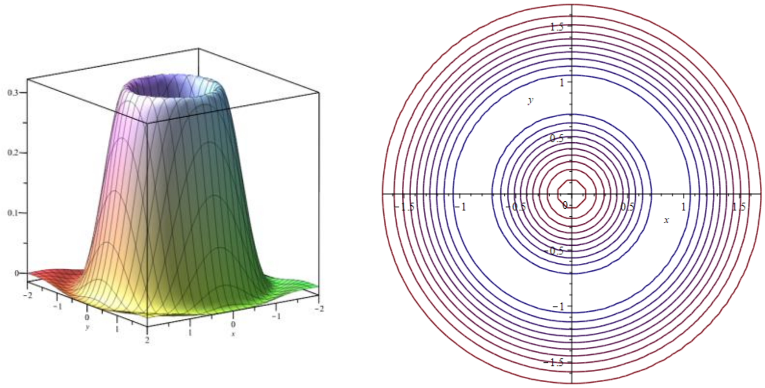


Complicated figures like this, are in practice displayed by CAS. Only simple figures may be sketched by hand.

4.

The geometry of shapes in nature is complicated, and it may probably not be possible to find useful equations, so level curves of the landscape are normally sketched as little lines/curves through measured points of the same altitude, for instance 41 meters above mean sea level.

Here we display a 3D figure with level curves viewed from above:



It is observed, that the figure is steep on the inside, shown by small distances between the level curves, - not steep on the top, - and steep again on the outside.

The main reason for displaying this figure is that we shall now consider the gradient. The gradient is biggest at close level curves.

The gradient

The word actually means slope, but here in 3D the meaning is expanded a bit. The gradient describes both the direction and the size of the slope (thus, it is a vector, - a lot more about this in Part 4).

$\frac{\partial z}{\partial x}$ shows the slope of the spacial figure in the x-direction

$\frac{\partial z}{\partial y}$ shows the slope of the spatial figure in the y-direction.

Just like before.

But what about the slope somewhere in between?

That is the gradient.

We consider slope in x and slope in y combined and write

$$\text{grad.}(z) = \text{grad.}(f(x,y)) = \begin{pmatrix} slope\ in\ x \\ slope\ in\ y \end{pmatrix} = \begin{pmatrix} partial\ diff.in\ x \\ partial\ diff.in\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix}$$

which is the definition of the gradient.

The size of the gradient is found by Pythagoras:

$$|\text{grad.}| = \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right)^{½}$$

Thus, the size alone is the slope with no knowledge of the direction.

We use Pythagoras because the slope in the x-direction is orthogonal to the slope in the y-direction.

This corresponds to a vector (where the combined direction is decided by the “strength ratio” of the two differential coefficients) and its length. The gradient is a vector. See more about vectors in Part 4.

Example 1

We display the function

$z = f(x,y) = x^2 + y^2$ which has

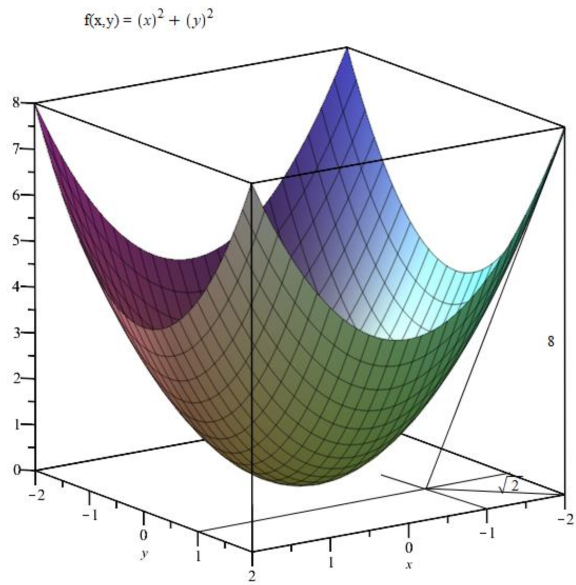
the slope in x $\frac{\partial z}{\partial x} = 2x$

the slope in y $\frac{\partial z}{\partial y} = 2y$ =>

The gradient = $\binom{2x}{2y}$ for instance x = |2| and y = |2| gives

the gradient = $\binom{4}{4}$ with the size

$|\text{grad.}| = \sqrt{4^2 + 4^2} \approx 5.66$



If we crawl up on the inside in the right corner where x = |2| and y = |2|, the slope thus is 5.66.

We may also regard it this way:

On the figure is sketched the tangent of the figure in the “corner” where x = |2| and y = |2|.

The helping lines show one step in x: $\Delta x = 1$

and one step in y: $\Delta y = 1$

which by Pythagoras gives a “common” step of $\sqrt{2}$.

In the height, the z-direction, we read $\Delta z = 8$

We calculate the “common” slope (= length of the gradient):

$|\text{grad.}| \; = \; \frac{8}{\sqrt{2}} \approx \; 5.66$ same answer.

Let us compare the slope when crawling up the ”corner” (which was calculated as 5.66) with the slope if we crawl 8 meters up in “the middle”:

In the middle we will come to the point:

z = 8, x = 0, y = ?

$z \; = \; x^2 + y^2 \quad \Leftrightarrow \quad y \; = \; 8^{½} \; \approx \; 2.83$

which is not visible on the figure. The point lies outside of the box shown.

gradient $= \binom{2x}{2y}$ which for x = |0| and y = |2.83| gives

gradient $= \binom{0}{5.66}$ with the size

$|\text{grad.}| \; = \; \sqrt{0^2 + \; 5.66^2} \; \approx \; 5.66$

Thus, the same slope, - as was expected for this rotation symmetrical figure.

The gradient may also be written with the symbol ∇

Part 4. Vectors

2D Vectors in the plane

A vector describes size and direction and is sketched as an arrow (long or short).

For instance it may be the size and direction of physical forces like strength and direction of the wind, strength and direction of sea currents and a lot more, - things that not just have a size (like a mass or an amount of money), but also a direction.

Vector mathematics is a tool for calculation. Therefore it is allowed to move a vector in the calculation, as long as it maintains length and direction. In math!

In physics and other fields, you *cannot* move a vector! It must not be moved away from the place of acting.

Vectors is a tool with corresponding calculation rules designed to enable and/or facilitate some calculations - particularly in 3D. Some of the methods may seem odd at first, though surely, they are useful. If we want to use the vector tool, we make them ourselves, and do calculations with them the way, we are about to explain.

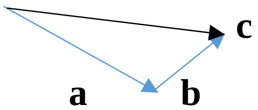
Two dimensional (2D) vectors do not enable us to do calculations we cannot do already, but they are necessary in 3D geometry. So, we build up the system in 2D and receive the reward in 3D.

Usually we call the vectors the same as the sides in a triangle, for instance $\vec{a}$ or $\overrightarrow{AB}$ - only with a small arrow on top. In other books it is maybe written $\overline{a}$ or $\overline{AB}$, and finally is used **a** or **AB.** We choose the latter.

We may add vectors, and subtract them. We may multiply and divide them with a constant, but we cannot multiply and divide them with one another. Yet here, the special tools the dot product and the determinant will apply, - more about this later.

Basics

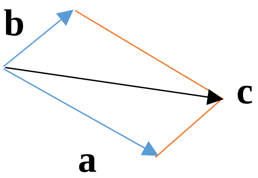
We can add vectors in two ways. One way is by putting them in extension of one another:



The black is the ”resultant”, here:

the sum vector **c**.

The other is to let them start at the same point and form a parallelogram:



The black is the ”resultant”, here:

the sum vector **c** (the same as before).

In order to differ from writing coordinates for points (in a row), vector coordinates are written in a column. We imagine that all vectors start at Origo, (x, y) = (0, 0), so that the vector coordinates, is the end point, the arrowhead. The vectors shown could be:

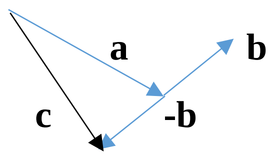
$$\mathbf{a} + \mathbf{b} = \mathbf{c} \quad \Rightarrow \quad \binom{5}{-3} + \binom{2}{2} = \binom{7}{-1}$$

It is also seen, that we add the x coordinates separately, and we add the y coordinates separately.

Or in letters for unknown coordinates:

a + **b** = **c** => $\binom{a_1}{a_2} + \binom{b_1}{b_2} = \binom{a_1+b_1}{a_2+b_2} = \binom{c_1}{c_2}$

We subtract a vector by adding the negative/opposite vector:



The black is the resultant **c**

a - b = c => $\binom{5}{-3} - \binom{2}{2} = \binom{3}{-5}$

Or in letters for unknown coordinates:

a - b = c => $\binom{a_1}{a_2}$ - $\binom{b_1}{b_2} = \binom{a_1-b_1}{a_2-b_2} = \binom{c_1}{c_2}$

We can multiply a vector by a constant (number or letter):

k $\cdot \binom{a_1}{a_2} = \binom{k \cdot a_1}{k \cdot a_2}$ or put out $\binom{k \cdot a_1}{k \cdot a_2} =$ k $\cdot \binom{a_1}{a_2}$

k may be everything (big, small, positive, negative) except 0. If k > 1 the vector becomes longer. If k < 1 the vector becomes shorter, - actually this is the same as dividing the vector by a number. If k is negative (k < 0) it will be directed oppositely.

Instead of multiplying/dividing vectors, the scalar-product was made/defined. It is often called the dot product, because a dot is used (similar to a multiplication dot). The technique is to multiply x by x and y by y, and finish by adding the two results:

$$\binom{5}{-3} \cdot \binom{2}{2} = 10 + (-6) = 4$$

So we start with vectors and yield a number.

Or with letters as unknown coordinates:

Dot product: $\mathbf{a} \cdot \mathbf{b} = \binom{a_1}{a_2} \cdot \binom{b_1}{b_2} = a_1b_1 + a_2b_2$ = a number

It turns out to be useful.

And now to something even more special: the determinant of two vectors. A determinant determines something for us, but first let us see the calculation technique:

The determinant: $\det(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1b_2 - a_2b_1$ = a number

We place vector a´s coordinates in the first column - and vector b´s coordinates in the second column. Then we multiply in a ”cross”: $a_1 \cdot b_2$ minus $a_2 \cdot b_1$ and yield a number as the answer.

So here, we also start with vectors and yield a number.

For fun we calculate the dot product (â is explained next page)

$$\hat{\mathbf{a}} \cdot \mathbf{b} = \binom{-a_2}{a_1} \cdot \binom{b_1}{b_2} = -a_2b_1 + a_1b_2 = a_1b_2 - a_2b_1 = \det(\mathbf{a}, \mathbf{b})$$

This also turns out to be useful.

We find a little more useful fun in this calculation:

$$-\det(\mathbf{b},\mathbf{a}) = -\begin{pmatrix} b_1 & a_1 \\ b_2 & a_2 \end{pmatrix} = -(b_1a_2 - b_2a_1) = a_1b_2 - a_2b_1 = \det(\mathbf{a}, \mathbf{b})$$

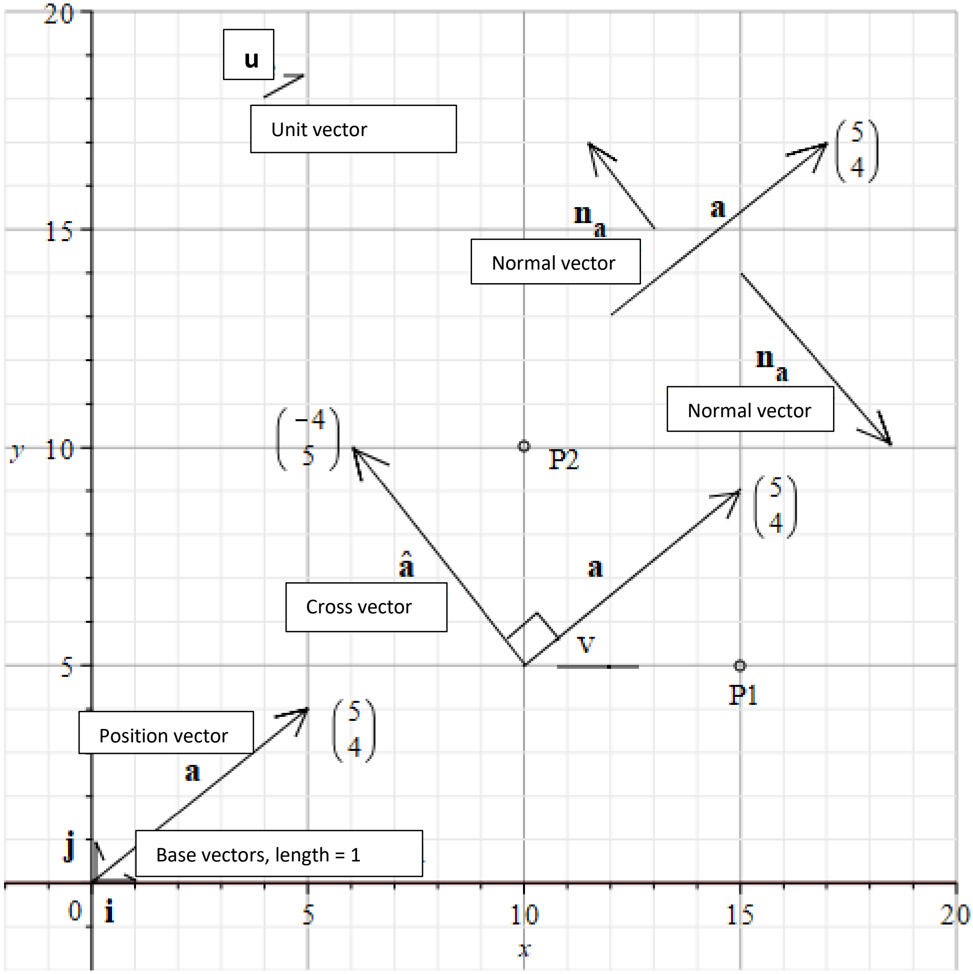
And now an old friend: We find the length of a vector by Pythagoras:

$|\mathbf{a}|^2 = x^2 + y^2$ or $|\mathbf{a}|^2 = a_1{}^2 + a_2{}^2$ =>

here: $|\mathbf{a}| = [5^2 + (-3)^2]^{½} = 34^{½}$

Special vectors

Some special vectors are shown in this diagram:



In mathematics, as mentioned, we imagine all vectors to start in Origo = (0, 0), so the vector coordinate is the end point - the arrowhead. Here **a** is shown in three places, but all three are the same vector, and it has the coordinates $\binom{5}{4}$. For those that do not start in Origo (0, 0), the coordinates are found by having: end minus start. For **a**:

$\binom{15-10}{9-5} = \binom{5}{4}$ and $\binom{17-12}{17-13} = \binom{5}{4}$

A vectors angle with the x-axis is

$v = \tan^{-1}\left(\frac{y}{x}\right)$ for **a**: $v = \tan^{-1}\left(\frac{4}{5}\right) \approx 38{,}7°$

If we turn a vector 90° positively (counter clock wise.), we get its cross-vector shown with a little hat. Its coordinates are inverted with a minus on the x coordinate. **a**'s cross-vector is

$\hat{\mathbf{a}} = \binom{-4}{5}$

It is seen by a helping point P1 that also turns 90°, and becomes P2. P1's x value of 5 becomes P2's y value of 5, and the y distance from P1 to vector, which is 4, becomes the x distance from P2 to the cross vector, which is -4.

Other vectors orthogonal to **a** are called normal vectors. Normal here means orthogonal. The normal vectors position, direction, and length, are not crucial if only the vector is orthogonal to our vector, it is a normal vector. There is an infinite number of normal vectors, but only one cross vector.

Up on the left is shown a vector with length 1. All vectors with length 1 are called unit vectors and we write $|\mathbf{u}| = 1$

Finally are shown two special unit vectors: **i** on the x-axis and **j** on the y-axis. They are sketched right next to the axis so that we can see them. They are called the base vectors.

In all mathematics we need zero. In vector mathematics we need a zero vector: **0**. If, for instance, we subtract **a** from **a** we get the zero vector:

a - a = 0

Examples

1.

A vector twice as long and in the opposite direction (-) has the coordinates:

$$-2 \cdot \binom{5}{4} = \binom{-10}{-8}$$

2.

Let us calculate the dot product of two orthogonal vectors, for instance a vector and its cross vector, here, **a** and **â** :

$$\binom{5}{4} \cdot \binom{-4}{5} = -20 + 20 = 0$$

In letters and multiplied by a constant, k, it is valid for all vectors:

$$\binom{h}{i} \cdot k\binom{-i}{h} = \binom{h}{i} \cdot \binom{k(-i)}{kh} = -hki + ikh = 0$$

This proves that a vector dotted with one of its normal vectors is 0.

3.

We will check if two walls are orthogonal. We form two vectors in the directions of the walls at some logical places: One vector leads from the corner of the walls to a window 3 meters away and has the coordinates in millimeters

$$\binom{0}{3000}$$

the other leads from the corner of the walls in the other direction to a door 1,76 meters away and has the coordinates in millimeter

$$\binom{1760}{10}$$

We check if the dot product yields 0:

$$\binom{0}{3000} \cdot \binom{1760}{10} = 0 + 30000 = 30000 \qquad \text{which is } \neq 0$$

No, the walls are not orthogonal. It is easy to see that the error is the 10 mm.

The two vectors we formed are also called direction vectors. They may have other lengths, as long as they are in the right direction.

4.

a may be split in two components one in the x-direction and another in the y-direction. It is written this way:

$$\mathbf{a} = 5\mathbf{i} + 4\mathbf{j} = \binom{5}{4}$$

So if we stand at Origo, walk 5 paces in x and 4 paces in y, we will be at the vectors end point (at the arrowhead).

Actually, we may split a vector in the directions we want, for instance:

$\binom{5}{4} = \binom{4}{2} + \binom{1}{2}$

Which is the deposit rule.

If **a** instead is called **OP** (because it goes from point O to point P) and we introduce a point Q in (4, 2) we have:

OP = **OQ** + **QP** = $\binom{4}{2} + \binom{1}{2} = \binom{5}{4}$

Or if we will find **QP**:

QP = OP - OQ = $\binom{5}{4} - \binom{4}{2} = \binom{1}{2}$

A vector that starts in O (0, 0) and leads to a known point (for instance P or Q) is also called a position vector because it leads from one position to another.

Calculation rules

There are four calculation rules for vectors. They resemble the standard rules of mathematics, only, we must remember that a dot between two vectors does not mean multiplication, it means dot.

1. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
2. $\mathbf{a}\cdot(\mathbf{b} + \mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
3. $(k\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(k\mathbf{b}) = k(\mathbf{a}\cdot\mathbf{b})$
4. $\mathbf{a}^2 = \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

Theorem 4 does not look like anything else so we will prove it. The right side

$\mathbf{a}^2 = \mathbf{a}\cdot\mathbf{a} = \binom{a_1}{a_2}\cdot\binom{a_1}{a_2} = a_1^2 + a_2^2$ and the left side

$|\mathbf{a}|^2 = a_1^2 + a_2^2$ gives the same

Angle

The Formula for the angle, v, between two vectors is found using the dot product or the determinant. The formula using the dot product is derived via the cosine rule:

$a^2 + b^2 - 2a\cdot b\cdot \cos C = c^2$ => here:

$|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}|\cdot|\mathbf{b}|\cdot \cos v = |\mathbf{a}\text{-}\mathbf{b}|^2$ where $|\mathbf{a}| = a$, $|\mathbf{b}| = b$, etc.

Rule 4 gives $|\mathbf{a}|^2 = \mathbf{a}^2$

Rule 4 gives $|\mathbf{b}|^2 = \mathbf{b}^2$

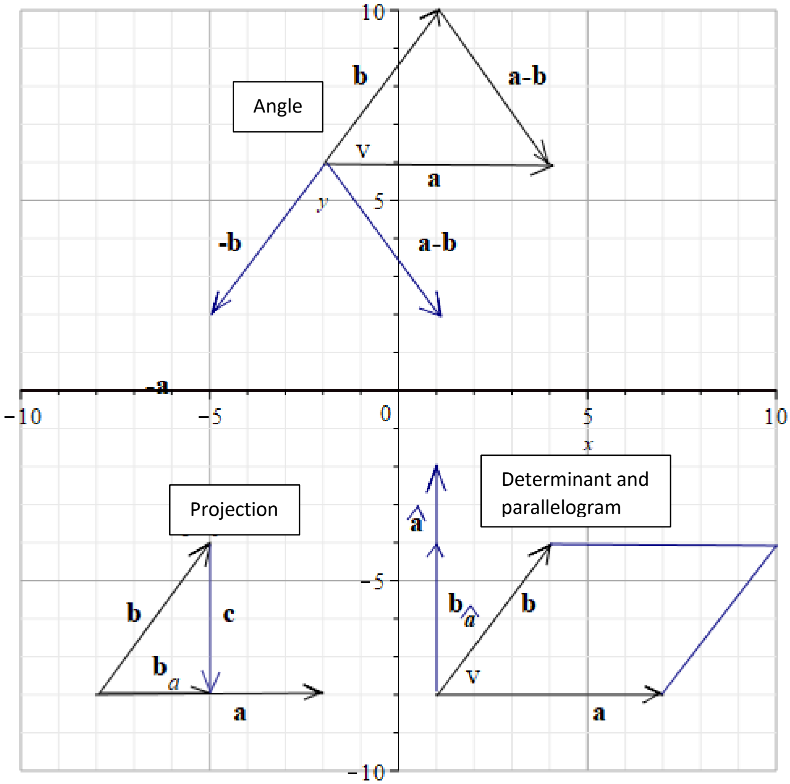
Rule 4 and 2 give $|\mathbf{a}\text{-}\mathbf{b}|^2 = (\mathbf{a}\text{-}\mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}\cdot\mathbf{b}$

Inserted $\mathbf{a}^2 + \mathbf{b}^2 - 2|\mathbf{a}|\cdot|\mathbf{b}|\cdot \cos v = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}\cdot\mathbf{b}$

Reduced $\cos v = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\boldsymbol{a}|\ |\mathbf{b}|}$

Which is the formula for the angle between two vectors. Also observe the uppermost figure in the diagram (next page) where it is shown that the opposite side to angle v is **a-b**.

Later we will present another formula that uses the determinant to calculate the angle between two vectors.



Projection

Also, in this diagram we show vector **b** projected in a straight angle onto vector **a**. The resultant is $\mathbf{b_a}$ and its coordinates depend on **b** and **a**, which are known.

We find vector $\mathbf{b_a}$'s coordinates by forming the blue helping vector **c** and then observe that:

$\mathbf{b_a} = k \cdot \mathbf{a}$ *(I)* $\quad\quad \mathbf{b_a} = \mathbf{b} + \mathbf{c}$ $\quad\quad$ and

$\mathbf{c} \cdot \mathbf{a} = 0$ $\quad\quad$ since they are orthogonal

Then, we will find k followed by finding $\mathbf{b_a}$:

$$0 = \mathbf{c} \cdot \mathbf{a} = (\mathbf{b_a} - \mathbf{b})\cdot\mathbf{a} = (k\mathbf{a} - \mathbf{b})\cdot\mathbf{a} = k\mathbf{a}^2 - \mathbf{a}\cdot\mathbf{b} = k|\mathbf{a}|^2 - \mathbf{a}\cdot\mathbf{b}$$

$$\Leftrightarrow \quad k|\mathbf{a}|^2 - \mathbf{a}\cdot\mathbf{b} = 0 \quad \Leftrightarrow \quad k|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{b} \quad \Leftrightarrow \quad k = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2} \quad \Rightarrow$$

k is inserted in *(I)* and yields:

$$\mathbf{b_a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2} \cdot \mathbf{a}$$

which is the formula for the coordinates of the projected vector.

Its length is found by the numerical value of the vectors:

$$|\mathbf{b_a}| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}|^2} \cdot |\mathbf{a}| \qquad \Leftrightarrow$$

$$|\mathbf{b_a}| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}|}$$

which is the projection-length-formula.

Determinant, area and angle

Now we shall see how to use the determinant:

In the low part of the diagram and to the right, is shown a parallelogram expanded by the vectors **a** and **b**. In a usual calculation the area is

$$\text{Area} = |\mathbf{a}| \cdot |\mathbf{b}_{\hat{\mathbf{a}}}|$$

We now have two ways to continue:

$$\text{Area} = |\mathbf{a}| \cdot |\mathbf{b}_{\hat{\mathbf{a}}}| = |\mathbf{a}| \cdot \frac{|\hat{\mathbf{a}}\cdot\mathbf{b}|}{|\hat{\mathbf{a}}|}$$ and since **a** and **â** are equally long

$$\text{Area} = |\hat{\mathbf{a}}\cdot\mathbf{b}|$$ which equals the determinant

$$\text{Area}_{\text{parallelogram}} = |\det(\mathbf{a}, \mathbf{b})|$$

So, we do not need to know $|\mathbf{b}_{\hat{\mathbf{a}}}|$ to find the area. We can find it directly from the vectors (**a** and **b**) that expand the parallelogram. .

We may also find the area by:

$|\mathbf{b}_{\hat{a}}| = |\mathbf{b}| \cdot \sin v$ =>

$\text{Area} = |\mathbf{a}| \cdot |\mathbf{b}_{\hat{a}}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin v$

or briefly

$\text{Area}_{\text{parallelogram}} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin v$

Since both methods renders the area, we deduce that

$\det(\mathbf{a}, \mathbf{b}) = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin v$ ⇔

$\sin v = \frac{\det(\mathbf{a},\mathbf{b})}{|\mathbf{a}| \cdot |\mathbf{b}|}$ which gives us another method of finding the angle between two vectors.

The first method is via the dot product

$\cos v = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$ which we found earlier.

There is more. Again, we consider the expression

$\det(\mathbf{a}, \mathbf{b}) = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin v$

which is 0 if v is 0. This means, that if the determinant is 0, **a** and **b** are parallel:

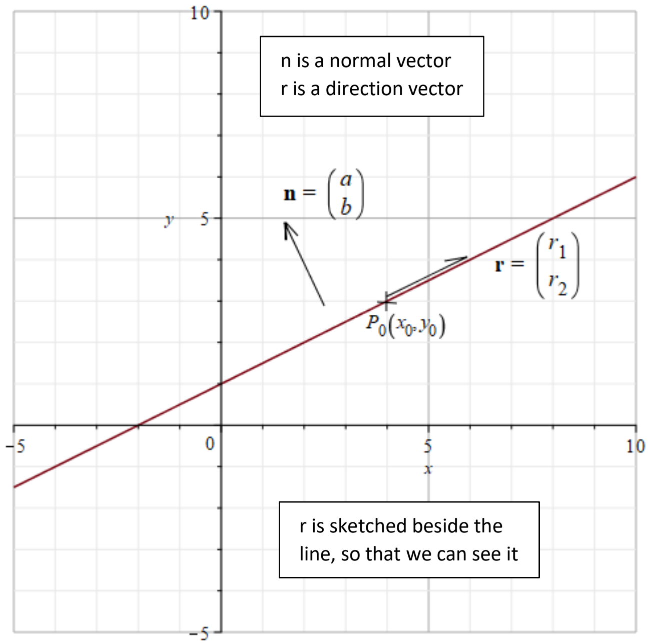
$\det(\mathbf{a}, \mathbf{b}) = 0 \quad \Leftrightarrow \quad \mathbf{a} \parallel \mathbf{b}$

Still there is more. We can also find the area of the triangle expanded by vector **a** and **b**:

$\text{Area}_{\text{triangle}} = \frac{1}{2} \cdot \text{Area}_{\text{parallelogram}} = \frac{1}{2} \cdot \det(\mathbf{a}, \mathbf{b}) = \frac{1}{2} \cdot |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin v$

The straight line on vector form

The equation for the straight line may also be written in two vector forms. Like before, we need two pieces of information about the line. We must know a point and a direction vector or a point and a normal vector.



Every known point on the line will do. Here we call the point P_0 with the coordinates (x_0, y_0), which will determine the line if we also know the direction.

The direction is described with a direction vector. Any direction vector will do (short, long, pointing forward or backwards, placed on the line or not) as long as it is parallel to the line.

The direction may actually also be determined by a normal vector (orthogonal to the line). Any normal vector will do.

If we imagine turning the line, the normal vector(s) will follow. Thus, each straight line has its own normal vector(s).

Using the direction vector, the straight line may be written as a vector function, which is rarely applied in 2D, but which is the only possibility in 3D (more later). This vector function is seen directly from the figure: Starting in P_0 the direction vector may move a point (the arrow head) up and down the line by multiplying with a parameter called t, which may be any number. "Parameter" is Greek and here means "along the measured".

$$\binom{x}{y} = \binom{x_0}{y_0} + t\binom{r1}{r2}$$ which is the 2D vector function of the straight line

By using the normal vector, the derivation is more complicated, but the result is more useful:

We observe from the figure

$$\mathbf{n} = \binom{a}{b}$$

If we form a cross vector of **n**, we get a direction vector for the line:

$$\mathbf{r} = \binom{-b}{a}$$

which we insert in the vector function of the straight line:

$$\binom{x}{y} = \binom{x_0}{y_0} + t\binom{-b}{a}$$

Here we split in an equation for the x-coordinate, and an equation for the y coordinate:

$x = x_0 + t(-b)$ and $y = y_0 + ta$

This is two equations with two unknowns. We isolate in the y-equation:

$t = \frac{y-y_0}{a}$ and insert into the x-equation:

$x = x_0 + \frac{y-y_0}{a} \cdot (-b)$ $\Leftrightarrow$

$x - x_0 = \frac{y-y_0}{a} \cdot (-b)$ ⇔

$a(x - x_0) = -b(y - y_0)$ ⇔

$a(x - x_0) + b(y - y_0) = 0$

This is the equation on vector form of the straight line. a and b are the coordinates of a normal vector, and (x_0, y_0) is a point on the line.

We may go on by multiplication into the parenthesis:

$ax - ax_0 + by - by_0 = 0$

and if we substitute $- ax_0 - by_0$ by c, we have:

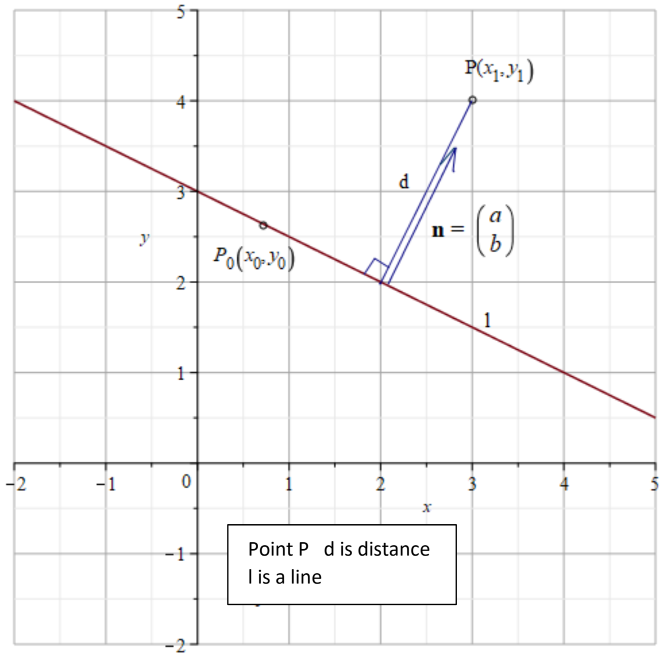
$ax + by + c = 0$

which is easier in some cases, and which is often listed in tables. We will soon use it in this form in the distance formula.

Thus, we have two equations for the straight line in ”ordinary” mathematics, and two (or three) equations within vector mathematics. Please note that a and b *do not* mean the same in the two systems.

Distance point-line

Vector mathematics can also be utilized to find the shortest distance (perpendicular) d from a point P to a line l.



For the derivation we need an arbitrary point on the line, which we call $P_0(x_0, y_0)$, and a normal vector for the line

$\mathbf{n} = \binom{a}{b}$

For simplicity we sketch the normal vector from l to P, - but any normal vector may be used.

We imagine a vector from P_0 to P (not sketched). If we project $\mathbf{P_0P}$ on to $\mathbf{n}$ we get a vector, d long. Then, applying the formerly derived projection formula:

$|\mathbf{b_a}| = \dfrac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}|}$ here

$d = \dfrac{|\mathbf{P_oP}\cdot\mathbf{n}|}{|\mathbf{n}|}$

numerator: $\binom{x_1 - x_0}{y_1 - y_0}\cdot\binom{a}{b} = ax_1 - ax_0 + by_1 - by_0$

P_0 is on l, so $-ax_0 - by_0$ is substituted by c, as before.

and denominator: $\sqrt{a^2 + b^2}$

inserted in the expression of d yields a formula for the distance between point and line (d for distance)

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

x_1 and y_1 are coordinates of the point, and a, b, and c are from the equation of the straight line in vector form.

Examples

1.

Let us find the angle between the two walls in a previous example, where the direction vectors were

$\binom{0}{3000}$ and $\binom{1760}{10}$ in millimeters

First, we have

$\cos v = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\boldsymbol{a}|\ |\mathbf{b}|}$ where

numerator $\binom{0}{3000} \cdot \binom{1760}{10} = 0 + 30{,}000 = 30{,}000$

denominator $(0^2 + 3000^2)^{½} \cdot (1760^2 + 10^2)^{½} = 5{,}280{,}085$

combined $v = \cos^{-1}\left(\dfrac{30{,}000}{5{,}280{,}085}\right) = 89.67°$

Which shows a small skewness.

Then in meters

$\binom{0}{3}$ and $\binom{1.760}{0.01}$

and now using

$\sin v = \frac{\det(\mathbf{a},\mathbf{b})}{|\mathbf{a}| \cdot |\mathbf{b}|}$ where

numerator $\begin{pmatrix} 0 & 1.76 \\ 3 & 0.01 \end{pmatrix} = 0 - 3 \cdot 1.76 = -5.28$

denominator $(0^2 + 3^2)^{½} \cdot (1.760^2 + 0.01^2)^{½} = 5.280085$

combined $v = \sin^{-1}\left(\frac{-5.28}{5.280085}\right) = (-)\ 89.67°$

Same angle as before. Naturally. The reason for the minus is the order of **a** and **b**, when calculating the determinant. At **b** before **a**, it would have been plus. Thus, we need to interfere and interpret the answer: The two formulas yield the same answer. If not so, we had made an error.

2.

We will project a vector $\mathbf{b} = \binom{1}{5}$ on a straight line with the equation $y = x + 3$

What will be the coordinates of the projected vector, and how long is it?

We use the projection formula, which has **b** projected on **a**:

$\mathbf{b_a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \cdot \mathbf{a}$

our **b** is $\binom{1}{5}$

our **a** we must form from a direction vector for the line. The slope is 1, so $\binom{1}{1}$ is a direction vector and is now our **a**:

numerator $\mathbf{a} \cdot \mathbf{b} = \binom{1}{1} \cdot \binom{1}{5} = 1 + 5 = 6$

denominator $|\mathbf{a}|^2 = ((1^2 + 1^2)^{½})^2 = 2$

Combined with **a** yields

$\mathbf{b_a} = \frac{6}{2}\binom{1}{1} = 3\binom{1}{1} = \binom{3}{3}$

Length

$|\mathbf{b_a}| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}|}$

numerator $|6| = 6$

denominator $(1^2 + 1^2)^{½} = \sqrt{2}$

combined $|\mathbf{b_a}| = \frac{6}{\sqrt{2}} \approx 4.24$

Controlling, we may use Pythagoras for $\mathbf{b_a}$'s coordinates:

$|\mathbf{b_a}| = (3^2 + 3^2)^{½} = \sqrt{18} \approx 4.24$ same answer

3.

A parallelogram is expanded by the vectors $\binom{5}{3}$ and $\binom{3}{4}$
What is the area?

$\text{Area}_{\text{parallelogram}} = \det(\mathbf{a}, \mathbf{b})$ =>

$A = \begin{pmatrix} 5 & 3 \\ 3 & 4 \end{pmatrix} = 5\cdot4 - 3\cdot3 = 20 - 9 = 11$

and if we change the order of the vectors:

$A = \begin{pmatrix} 3 & 5 \\ 4 & 3 \end{pmatrix} = 3\cdot3 - 4\cdot5 = 9 - 20 = -11$

Here we must interfere, since an area can only be positive

$|A| = 11$

Then we have the same answer.

4.

A triangle is expanded by the vectors $\binom{5}{3}$ and $\binom{3}{4}$
What is the area?

$\text{Area}_{\text{triangle}} = \frac{1}{2} \cdot \det(\mathbf{a}, \mathbf{b})$ =>

$A = \frac{1}{2} \cdot \begin{pmatrix} 5 & 3 \\ 3 & 4 \end{pmatrix} = \frac{1}{2} \cdot (5 \cdot 4 - 3 \cdot 3) = \frac{11}{2} = 5{,}5$

5.

A straight line passes through point (6,8) and has a normal vector $\binom{3}{4}$ What is the lines equation?

$a(x - x_0) + b(y - y_0) = 0$ =>

$3(x - 6) + 4(y - 8) = 0$ ⇔

$3x - 18 + 4y - 32 = 0$ ⇔

$3x + 4y - 50 = 0$ which is the answer

The equation in "ordinary" mathematics will be:

$y = ax + b$

where a now is the slope, and b now is the y-value where the line intersects the y-axis!

$3x + 4y - 50 = 0$ ⇔

$4y = -3x + 50$ ⇔

$y = -\frac{3}{4}x + \frac{25}{2}$

Thus, a line with the slope $-\frac{3}{4}$ and with intersecting the y-axis in point $(0, \frac{25}{2})$.

6.

What is the distance between point (2,1) and line $y = -\frac{3}{4}x + \frac{25}{2}$?

We need the line on vector form, so we change it

$y = -\frac{3}{4}x + \frac{25}{2} \quad \Leftrightarrow$

$4y = -3x + 50 \quad \Leftrightarrow$

$3x + 4y - 50 = 0$

and use the distance formula

$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \quad =>$

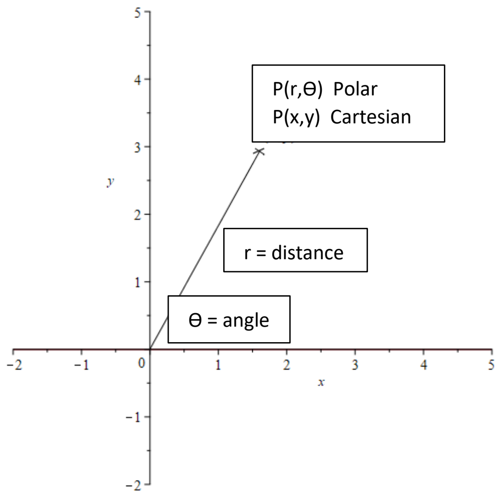
$d = \frac{|3 \cdot 2 + 4 \cdot 1 - 50|}{\sqrt{3^2 + 4^2}} \quad \Leftrightarrow$

$d = \frac{|-40|}{\sqrt{25}} \quad \Leftrightarrow$

$d = 8$

Polar coordinates in 2D

Coordinates may also be specified as polar coordinates which is: (distance from Origo , angle with the +x axis), see the figure:



This may be an advance in aviation techniques, etc., particularly when we expand to 3D.

Conversion between Cartesian (ordinary) coordinates, position vector coordinates and polar coordinates is shown in this table:

	Coordinates	Length/distance
Cartesian	P(x,y)	$r = (x^2 + y^2)^{½}$
Pos. vector	$\mathbf{OP} = \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cdot\cos\Theta \\ r\cdot\sin\Theta \end{pmatrix}$	$\lvert\mathbf{r}\rvert = \lvert\mathbf{OP}\rvert = (x^2 + y^2)^{½}$
Polar	P(r,Θ)	$r = [(r\cdot\cos\Theta)^2 + (r\cdot\sin\Theta)^2]^{½}$

It appears that length/distance is Pythagoras in all three cases.

Polar coordinates are referred to in other literature, yet they will be considered a little more, when dealing with complex numbers at the end of this book.

Vector functions (parametric curves) in 2D

If we consider a function that goes forth and back in the x-direction (for instance a circle function), there is no longer just one y-value for each x-value. Then we must describe it as a vector function, and its curve in a coordinate system is called a parametric curve. We introduce a parameter, usually called t, since the parameter is often time or proportional with time. Thus, the parameter defines where we are on the curve.

We write the vector function this way

$\mathbf{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$ which is the definition of a vector function

A vector function in 2D may be seen as *one* equation describing x as a function of t, and *another* equation that describes y as a function of t.

The vector function for a straight line

Let us consider an example we know already, namely the straight line

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ the vector function of a straight line in 2D $\Leftrightarrow$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0+t\cdot r_1 \\ y_0+t\cdot r_2 \end{pmatrix}$

or as two equations

$x(t) = x_0 + t\cdot r_1$ and $y(t) = y_0 + t\cdot r_2$

A straight line does not go forth and back in the x-direction, so we do not really need this vector function but it is a fine example.

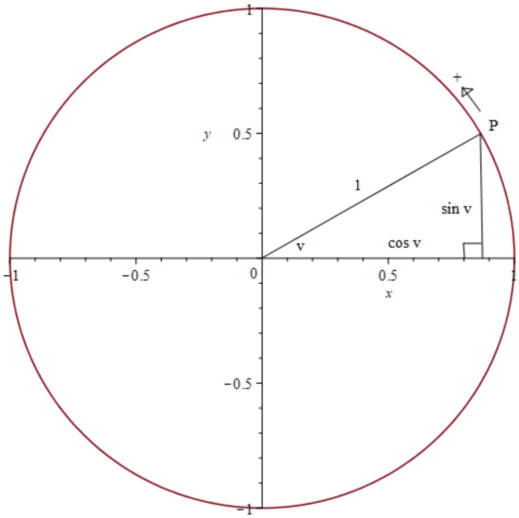
The vector function of a circle

The circles equation is known to us

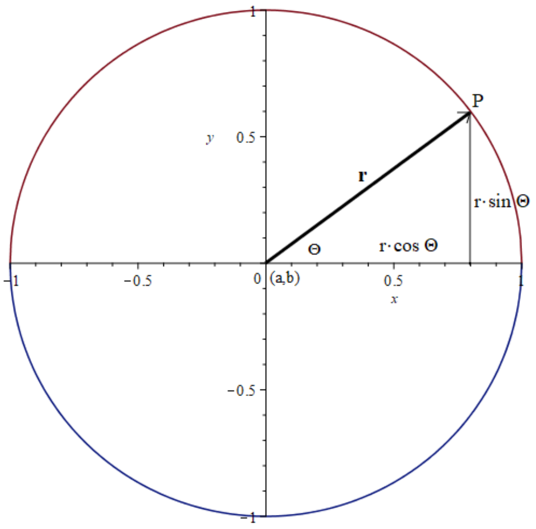
$(x - a)^2 + (y - b)^2 = r^2$

which we found by Pythagoras, and which gives us a still image of the coordinates (x,y) for a point on the circle. But if, for instance, we know a, b, y and r, and isolate x, we have a second degree equation with two roots for x. If we want just one root, we have to find the circles vector function:

Again, we consider the unit circle



and change the symbols



Now, Origo is the local centre of a non-rotating (still) machine placed in a building. Relative to the coordinate system of the building our Origo has the coordinates O(a,b).

The point P (a painted dot on the rotor) has the distance r (radius) from the centre (O) and is described by the position vector (also called radius vector), **r**.

The angle v, measured in degrees, is now the angle Θ (the Greek letter teta) measured in radians.

Relative to the *machines own local coordinate system*, P has the coordinates

P(r·cos Θ, r·sin Θ)

and the vector function with position vector **r** becomes

$\mathbf{r}(\Theta) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cdot\cos\theta \\ r\cdot\sin\theta \end{pmatrix}$ now Θ is the parameter

Relative to the *buildings coordinate system*, **r** has the coordinates

$\mathbf{r}(\Theta) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a + r\cdot\cos\theta \\ b + r\cdot\sin\theta \end{pmatrix}$

which is the circle's fully expanded vector function (or parametric function).

Please note that now Θ is the variable. The position **r** depends on the angle (the parameter) Θ.

Differentiation of vector functions

The parametric curves of vector functions have tangents. The slope of a tangent is found by differentiating as we are used to following the same calculation rules. The novelty is to split into an equation for x and an equation for y, which are differentiated separately with respect to t. So: how does x change when t changes? and how does y change, when t changes? We observe this by the differential coefficient

$$\mathbf{r}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$$

Vector functions may have *vertical* tangents.

Differentiation of the vector function of a straight line

Let us again look at the example we know already, the straight line. We know that the differential coefficient shall render the constant slope, a. Let us see:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \qquad \Rightarrow$$

$x(t) = x_0 + t \cdot r_1$ and $y(t) = y_0 + t \cdot r_2$ =>

$x'(t) = 0 + r_1$ and $y'(t) = 0 + r_2$ =>

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$ which is a direction vector giving the slope:

$$\text{slope} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)} = \frac{r_2}{r_1} = a$$

Thus, when we differentiate the vector function of the line, we can get the slope, which we usually call a. So, it corresponds with what we already know.

Differentiation of the vector function of the circle

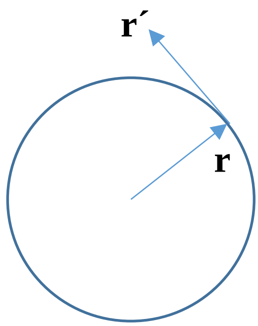
$\mathbf{r}(\Theta) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a + r \cdot \cos\theta \\ b + r \cdot \sin\theta \end{pmatrix}$ Θ is the parameter =>

We differentiate with respect to Θ and get

$$\mathbf{r}'(\Theta) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -r \cdot \sin\theta \\ r \cdot \cos\theta \end{pmatrix}$$

We see that (a,b) disappears (which corresponds to the fact, that the position of the circle in the building surely has no influence on the tangent slope of the circle). Also, we see that r´ has an x-coordinate similar to the y-coordinate for r, only opposite (minus), and that r´ has a y-coordinate similar to the x-coordinate for r. Thus, r´ is turned 90° relative to r (which corresponds with the direction of the tangent).

Displayed in a simplified figure:

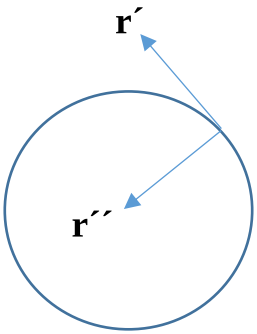


If we differentiate once more (the second order differential coefficient, the second order derivative) we get

$$\mathbf{r}''(\Theta) = \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -\,r\cdot\cos\theta \\ -\,r\cdot\sin\theta \end{pmatrix}$$

which is a vector **r´´** directed opposite (due to minus for both x and y) to **r**.

Displayed in a simple figure:



The first derivative is tangent to the circle, as expected.

The second derivative has no immediate significance in mathematics, but it does in physics, as can be seen in the following example.

Example 1

Now the machine rotates with a constant speed (constant angular velocity) and we watch the point P (the painted dot).

Furthermore, we define the constant angular velocity:

$$\text{constant angular velocity} = \frac{\textit{angle turned in radians}}{\textit{time in seconds}} \qquad \Rightarrow$$

$$\omega = \frac{\theta}{t} \qquad \Leftrightarrow \qquad \Theta = \omega t$$

Since ω is constant, the variable changes from Θ to t, and the three equations are

$$\mathbf{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a + r\cdot\cos \omega t \\ b + r\cdot\sin \omega t \end{pmatrix} \qquad =>$$

$$\mathbf{r}'(t) = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -r\omega\cdot\sin \omega t \\ r\omega\cdot\cos \omega t \end{pmatrix} \qquad \text{diff. “outer, inner”} \qquad =>$$

$$\mathbf{r}''(t) = \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -r\omega^2\cdot\cos \omega t \\ -r\omega^2\cdot\sin \omega t \end{pmatrix} \qquad \text{diff. “outer, inner”}$$

r is the position

r´ is called the tangential velocity $\mathbf{v}_{tan}$

r´´ is called the centripetal acceleration $\mathbf{a}_c$

As observed, the centripetal acceleration is directed toward the centre of the circle, which is the case for all circular motions with a constant speed.

Circular motion with variable speed also have a centripetal acceleration directed toward the centre (otherwise there would be no circular motion), - which, with a tangential acceleration comprise the whole acceleration (two components).

2.

Let us try to find the formulas for the size of v_{tan} and a_c:

Pythagoras for v_{tan}

$$v_{tan} = [(-r\omega\cdot\sin \omega t)^2 + (r\omega\cdot\cos \omega t)^2]^{½} \qquad \Leftrightarrow$$

$$v_{tan} = [(r^2\omega^2((\sin \omega t)^2 + (\cos \omega t)^2)]^{½} \qquad \Leftrightarrow$$

and by the base relation: $(\sin v)^2 + (\cos v)^2 = 1^2$

it all becomes much shorter: =>

$v_{tan} = \omega r$ which is the formula for the size of the tangential velocity

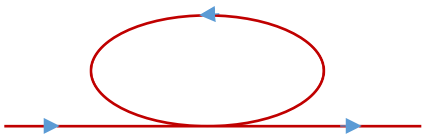
Pythagoras for a_c

$a_c = [(- r\omega^2 \cdot \cos \omega t)^2 + (- r\omega^2 \cdot \sin \omega t)^2]^{½}$ $\Leftrightarrow$

$a_c = [(r^2\omega^4((\sin \omega t)^2 + (\cos \omega t)^2)]^{½}$ $\Leftrightarrow$ base relation

$a_c = r\omega^2$ which is the formula for the size of the centripetal acceleration

Double points



If the curve of a vector function intersects itself, we have a double point. It is easier to find the double point by sketching the curve in a diagram and read the coordinates, i.e. a graphic solution. A task for CAS.

The definition of a double point is

$\mathbf{r}(t_1) = \mathbf{r}(t_2)$ => $\begin{pmatrix} x(t_1) \\ y(t_1) \end{pmatrix} = \begin{pmatrix} x(t_2) \\ y(t_2) \end{pmatrix}$

A double point has the same values of x, and the same values of y. The difference is t.

Example

We will investigate if there is/are double point(s), and find the coordinates, in the vector function

$\mathbf{r}(t) = \begin{pmatrix} t^3 - t \\ t^2 - 1 \end{pmatrix}$ =>

$\mathbf{r}(t_1) = \begin{pmatrix} t_1{}^3 - t_1 \\ t_1{}^2 - 1 \end{pmatrix}$ $\mathbf{r}(t_2) = \begin{pmatrix} t_2{}^3 - t_2 \\ t_2{}^2 - 1 \end{pmatrix}$

In the double point we have

$x_1 = x_2$ => $t_1^3 - t_1 = t_2^3 - t_2$ *equation 1* and

$y_1 = y_2$ => $t_1^2 - 1 = t_2^2 - 1$ *equation 2*

Thus, two equations with two unknowns. If we solve by CAS we have $t_1 = \pm 1$ and $t_2 = \pm 1$

So we get to the double point at $t = -1$ and again at $t = 1$

The equations are difficult to solve manually, but we will try:

2. $t_1^2 - 1 = t_2^2 - 1$ ⇔ $t_1^2 = t_2^2$

At first $t_1 = t_2$ but that is false, since the roots must be different. The only true answer is:

$t_1 = -t_2$

which we insert into *equation 1:*

1. $t_1^3 - t_1 = t_2^3 - t_2$ => $(-t_2)^3 + t_2 = t_2^3 - t_2$ ⇔

$-t_2^3 - t_2^3 = -t_2 - t_2$ => $-2t_2^3 = -2t_2$ ⇔

$t_2^3 = t_2$

which is a third degree equation with up to three roots, found by guessing: 0 is ok, 1 is ok, -1 is ok. Inserted into *equation 2:*

2. $t_2 = 0$ => $t_1 = 0$ $t_2 = 1$ => $t_1 = -1$ $t_2 = -1$ => $t_1 = 1$

Since t_1 and t_2 must be different, 0 is no root. Left are the roots:
$t_1 = \pm 1$ and $t_2 = \pm 1$.

Same answer as using CAS.

Thus we get to the double point at t = -1 and again at t =1

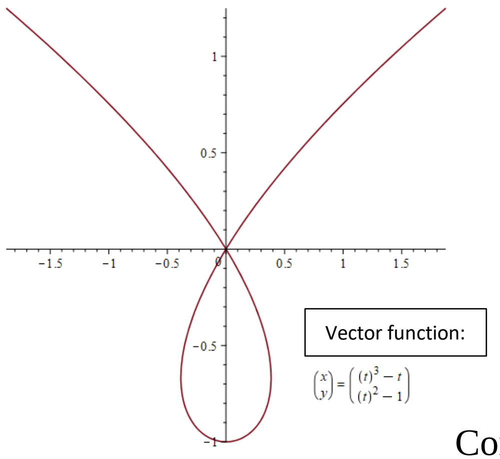
Then we find the x and y coordinates of the double point:

We insert t = 1 => $\mathbf{r}(t_1) = \binom{0}{0}$ and $\mathbf{r}(t_2) = \binom{0}{0}$

and insert t = -1 => $\mathbf{r}(t_1) = \binom{0}{0}$ and $\mathbf{r}(t_2) = \binom{0}{0}$

Thus, one double point with the coordinates (x, y) = (0, 0)

Let us finish the example by displaying the curve in a diagram:



Compliance.

Vector functions may have horizontal and/or vertical tangent(s) for:

horizontal: $\frac{dy}{dt} = y'(t) = 0$ and vertical: $\frac{dx}{dt} = x'(t) = 0$

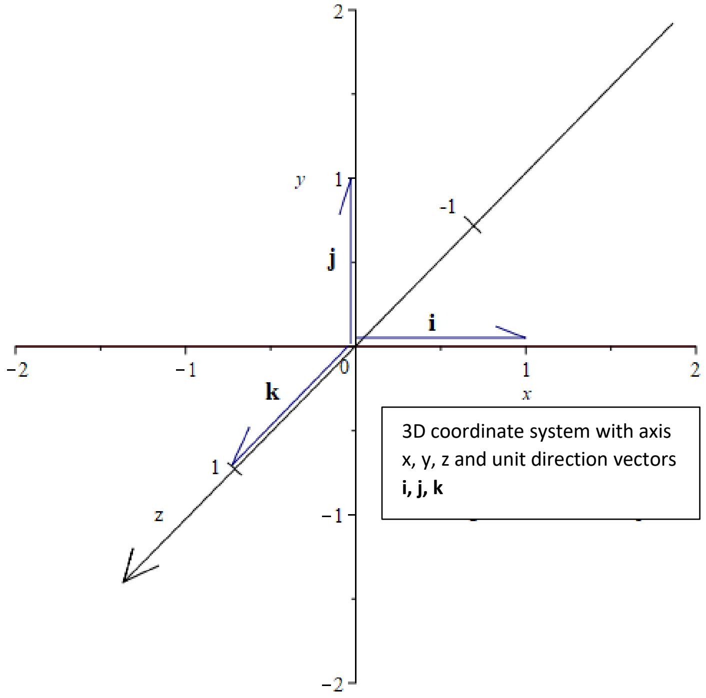
In the example:

Horizontal tangent in point: $y'(t) = 2t = 0$ => $t = 0$ => $(x,y) = (0,-1)$

Vertical: $x'(t) = 3t^2 - 1 = 0$ => $t = \pm 0.58$ => $(x,y) = (0.38\ ;\ -0.67)$ and $(-0.38\ ;\ -0.67)$

3D Vectors in the space

The vector tool is very useful working in 3D, in the space. Here with the z-axis (the third axis) out from the plane of the paper:



The unit base vectors are sketched slightly beside the axis, so that we can see them.

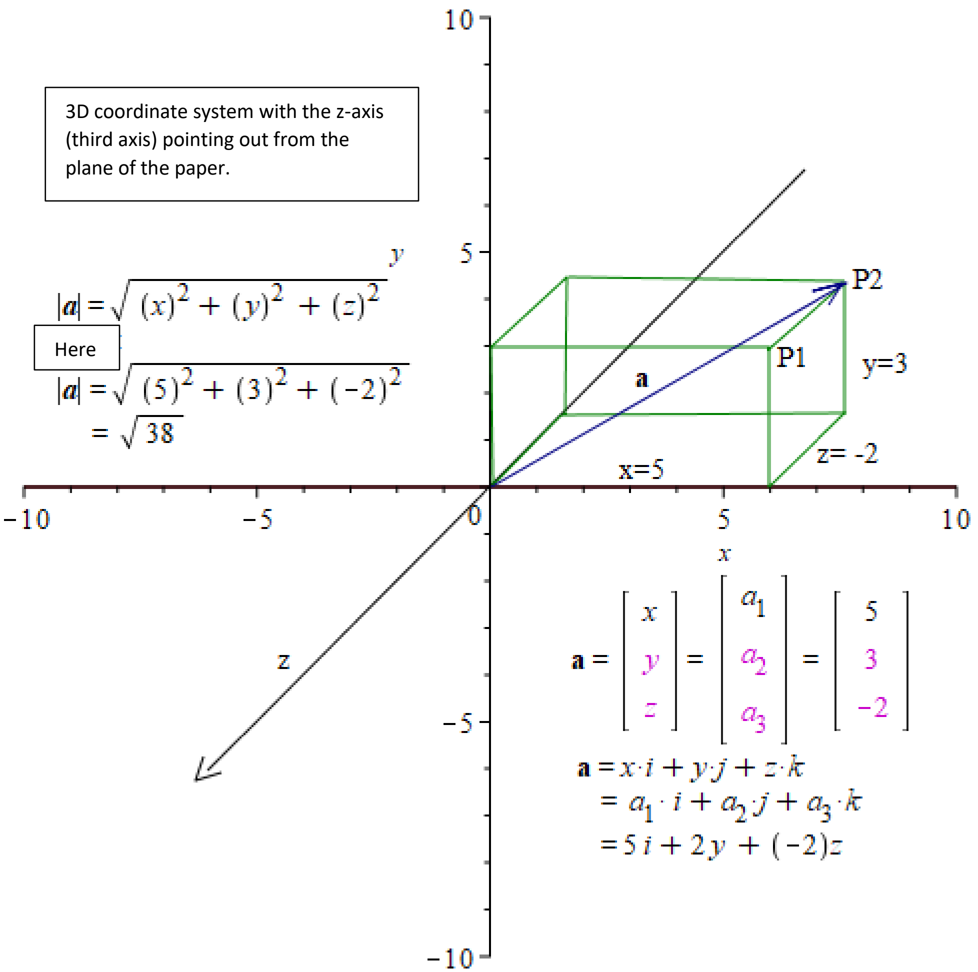
The coordinate system may be sketched in other positions, as long as the order is x, y, z in the positive direction (counter clock wise).

Most formulas are the same as for 2D. They only need to be expanded by the third coordinate we call z, - then it is 3D. The calculation technique is also the same.

The differences are:

- The determinant does not exist in 3D
- The cross vector is not in use in 3D.
- A normal vector may now be found from a novel, a little odd, tool called the cross product or the vector product.

In the diagram, we display a position vector **a** in 3D in a coordinate system.



We also see a 3D version of Pythagoras, with the z-coordinate added. It derives this way:

a may be split in a vector from O (Origo) to P1 with the length $(x^2 + y^2)^{½}$ plus a vector in the z-direction from P1 to P2 with the length z. These two vectors are orthogonal, so Pythagoras applies:

$|\mathbf{a}|^2 = [(x^2 + y^2)^{½}]^2 + z^2 \quad =>$

$|\mathbf{a}|^2 = (x^2 + y^2) + z^2 \quad \Leftrightarrow$

$|\mathbf{a}|^2 = x^2 + y^2 + z^2 \quad \Leftrightarrow$

$|\mathbf{a}| = (x^2 + y^2 + z^2)^{½}$

Which also may be written as a square root, as shown in the diagram.

Distance point-point

The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is also found from Pythagoras.

$|P_1P_2| = ((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2)^{½}$

Examples

1.

We have two vectors $\mathbf{a} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ -6 \\ 7 \end{pmatrix}$

Their sum is $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ -6 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$

and difference $\quad \mathbf{a} - \mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ -6 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ -9 \end{pmatrix}$

and $\quad \mathbf{b} - \mathbf{a} = \begin{pmatrix} -1 \\ -6 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -9 \\ 9 \end{pmatrix}$

x coordinates alone, y alone, and z alone.

2.

A vector twice (2) as long as **a** in the diagram and in the opposite direction (-) has the coordinates

$$-2 \cdot \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ 4 \end{pmatrix} = -2\mathbf{a}$$

3.

Let us calculate the dot product of two 3D vectors, for instance **a** and **-2a**

$$\mathbf{a} \cdot (-2)\mathbf{a} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ -6 \\ 4 \end{pmatrix} = (-50) + (-18) + (-8) = -76$$

4.

a may be split in three components, one in the x direction, one in y, and one in z. We write it this way:

$$\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} + (-2)\mathbf{k} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$$

So if we stand at Origo and walk 5 paces in x, 3 paces in y, and 2 paces in -z, we will be at the vectors end point (arrowhead).

Also in 3D, we may split a vector in the components you want to, for instance:

$$\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix}$$

The deposit rule.

If, instead of **a**, we use the name **OP** (because it leads from point O to point P) and introduce a point Q in (1, 0, 2) , we have:

$$\mathbf{OP} = \mathbf{OQ} + \mathbf{QP} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$$

We go from O to Q and on to P. Combined from O to P.

Or if we will find **QP**:

$$\mathbf{QP} = \mathbf{OP} - \mathbf{OQ} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix}$$

5.

The distance between two points A(1, -1, 8) and B(-2, 3, -3) is

$|P_1P_2| = ((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2)^{½}$ => here

$|AB| = ((-2) - 1)^2 + (3 - (-1))^2 + ((-3) - 8)^2)^{½} \approx 12{,}08$

More theory

The cross product (the vector product)

The cross product of two vectors is written this way:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} d_{23} \\ d_{31} \\ d_{12} \end{pmatrix}$$

d for determinant.

and is calculated by putting 3 determinants on top of each other and multiply in a "cross" (just like the determinant in 2D):

$\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = a_2b_3 - a_3b_2 = d_{23} =$ a number for the new vectors x value

$\begin{pmatrix} a_3 & b_3 \\ a_1 & b_1 \end{pmatrix} = a_3b_1 - a_1b_3 = d_{31} =$ a number for the new vectors y value

$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1b_2 - a_2b_1 = d_{12} =$ a number for the new vectors z value

Rendering a new vector with the calculated coordinates.

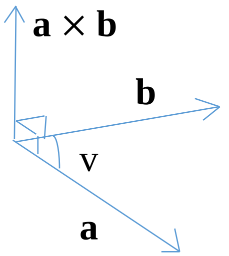
Peculiar, but as for the dot product and the 2D-determinant it turns out to be useful.

Namely, it turns out that the cross vector is orthogonal on both of the original vectors, here: **a** and **b**. This we find, because the dot products are zero:

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} =$$

$(a_1a_2b_3 - a_1a_3b_2) + (a_2a_3b_1 - a_2a_1b_3) + (a_3a_1b_2 - a_3a_2b_1) = 0$

the dot product is zero meaning **a** and the cross vector **(a × b)** is orthogonal. A similar calculation shows that **b** and **(a × b)** are orthogonal too. Thus, the cross vector is a normal vector to both **a** and **b**.



(**b** × **a** would give the opposite cross vector)

Thus, we have a tool to find a normal vector for **a** and **b**, which turns out to be crucial in the coming formulas.

The angle between two vectors

Just like calculating the angle between two vectors using the dot product

$\cos v = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|}$ proved in 2D, also valid in 3D

and the determinant from 2D

$\sin v = \dfrac{\det(\mathbf{a},\mathbf{b})}{|\mathbf{a}| \cdot |\mathbf{b}|}$ proved in 2D, only valid in 2D

we may find the angle using the cross product in 3D

$\sin v = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| \cdot |\mathbf{b}|}$ to be shown with numbers in 3D

It appears, that the determinant in 2D has become the cross product in 3D. A proof in letters for this formula is very long. Instead we show it with numbers in a following example.

Area

We can also find the area of the parallelogram expanded by the two vectors **a** and **b**. We know from 2D:

$$\text{Area}_{\text{parallelogram}} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin v$$

If we compare this equation with the formula above in the form

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin v$$

we find the area of the parallelogram the vectors expand

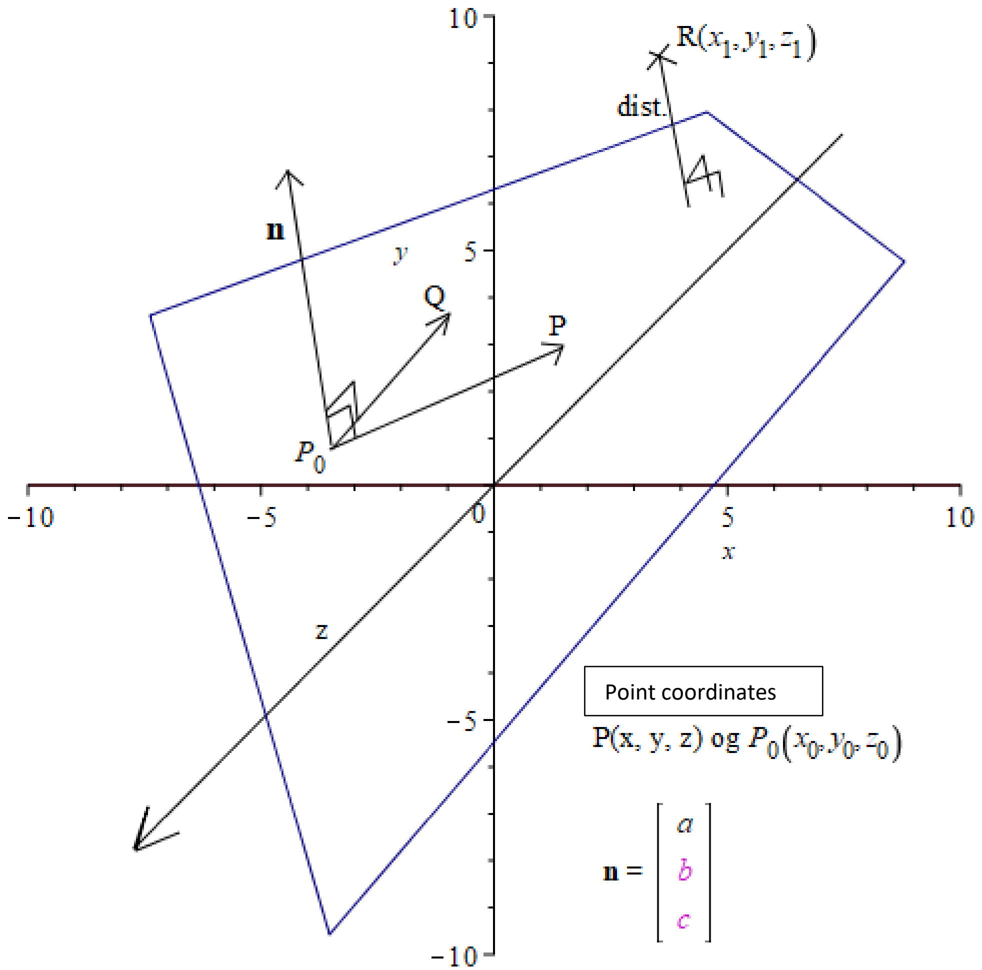
$$\text{Area}_{\text{parallelogram}} = |\mathbf{a} \times \mathbf{b}|$$

and for the triangle the vectors expand

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \cdot |\mathbf{a} \times \mathbf{b}|$$

The equation of the plane

A plane may be a wall (straight or oblique), a floor, a roof side, etc. We need to know the equation of a plane. A plane is a 2D figure, but it may have an oblique position in a 3D coordinate system and must consequently have a 3D formula. A plane is infinite in its two directions, but its image may be limited:



We must know three things about a plane, usually three points. They are used to form two vectors and their cross vector, which then is a normal vector of the plane.

Try to hold a book, and put a pencil orthogonally to the front or back (that does not matter). Imagine the pencil sits firmly. If the book is turned, the pencil will follow. So the normal vector ”belongs” to the plane and may be used in the planes equation:

In the figure, the known points are used to form two vectors $\mathbf{P_0P}$ and $\mathbf{P_0Q}$. Their cross product gives a normal vector $\mathbf{n}$. We know that two orthogonal vectors have a dot product of 0

$\mathbf{P_0Q} \cdot \mathbf{n} = 0$ =>

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \Leftrightarrow$$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

which is the planes equation, where a, b, c are the coordinates for a normal vector of the plane, and x_0, y_0, z_0 are the coordinates of a point in the plane.

If we multiply into the parenthesis

$ax - ax_0 + by - by_0 + cz - cz_0 = 0$ =>

Since P_0 is a known point in the plane, P_0's coordinates will fulfil the planes equation. Therefore we regard $-ax_0 - by_0 - cz_0$ as a known size called d. Thus

$ax + by + cz + d = 0$

is a short version of the plane's equation.

Distance point-plane

In the diagram is also shown a point R with the perpendicular distance dist. to the plane.

The distance formula point-plane is derived like this:

We form a vector $\mathbf{P_0R}$ from the two points in the diagram (the vector is not sketched)

$$\mathbf{P_0R} = \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{pmatrix}$$

which we project onto the normal vector of the plane

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

by using the projection-length-formula

$$|\mathbf{b_a}| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{a}|}$$

which in our case is

$$\text{dist.} = \frac{|\mathbf{P_oR} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

The numerator is

$$\left| \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| = |\, ax_1 - ax_0 + by_1 - by_0 + cz_1 - cz_0 \,|$$

as before we substitute $-ax_0 - by_0 - cz_0 = d$ =>

$|\, ax_1 + by_1 + cz_1 + d \,|$

The denominator is

$\sqrt{a^2 + b^2 + c^2}$ (Pythagoras)

Combined we have

$$\text{dist.} = \frac{|\, ax_1 + by_1 + cz_1 + d \,|}{\sqrt{a^2 + b^2 + c^2}}$$

where x_1, y_1, z_1 are the coordinates of the point, - and a, b, c, d come from the short version of the plane's equation.

Examples

1.

Find the angle between two vectors and the area of the parallelogram they expand, when

a = $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and **b** = $\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

The angle is calculated via the dot product in

$$\cos v = \frac{\mathbf{a} \cdot \mathbf{b}}{|\boldsymbol{a}|\,|\mathbf{b}|}$$

where the numerator $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} = -1 - 4 + 9 = 4$

and the denominator $(1^2 + 2^2 + 3^2)^{½} \cdot ((-1)^2 + (-2)^2 + 3^2)^{½} = 14$

Combined $\cos v = \frac{4}{14} \Leftrightarrow v = \cos^{-1}\left(\frac{4}{14}\right) \approx 73.4°$

Let us see the same using the cross product

$$\sin v = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| \cdot |\mathbf{b}|}$$

numerator $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 - 3 \cdot (-2) \\ 3 \cdot (-1) - 1 \cdot 3 \\ 1 \cdot (-2) - 2 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \\ 0 \end{pmatrix}$

numerator $\left| \begin{pmatrix} 12 \\ -6 \\ 0 \end{pmatrix} \right| = (12^2 + (-6)^2 + 0^2)^{½} = 180^{½} = \sqrt{180}$

denominator $(1^2 + 2^2 + 3^2)^{½} \cdot ((-1)^2 + (-2)^2 + 3^2)^{½} = 14$

Combined $\quad \sin v = \frac{\sqrt{180}}{14} \Leftrightarrow v = \sin^{-1}\left(\frac{\sqrt{180}}{14}\right) \approx 73.4°$

Same answer.

The area of the parallelogram the vectors expan, we chose to calculate by

$\text{Area}_{\text{parallelogram}} = |\mathbf{a} \times \mathbf{b}|$ => here

$\text{Area}_{\text{parallelogram}} = \sqrt{180} \approx 13.4$

since the cross product was calculated above.

2.

Let us find the equation of a plane with a normal vector

$\mathbf{n} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$ and a point $P_0(4, 3, -5)$ =>

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ => here

$-1(x - 4) + 5(y - 3) + 2(z - (-5)) = 0 \Leftrightarrow$

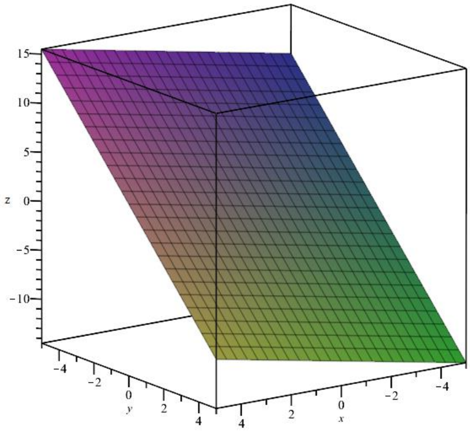
$-x + 4 + 5y - 15 + 2z + 10 = 0 \Leftrightarrow$

$-x + 5y + 2z - 1 = 0 \Leftrightarrow$

$x - 5y - 2z + 1 = 0$

which is the equation of our plane.

Here we display this plane in a 3D plot:



3.

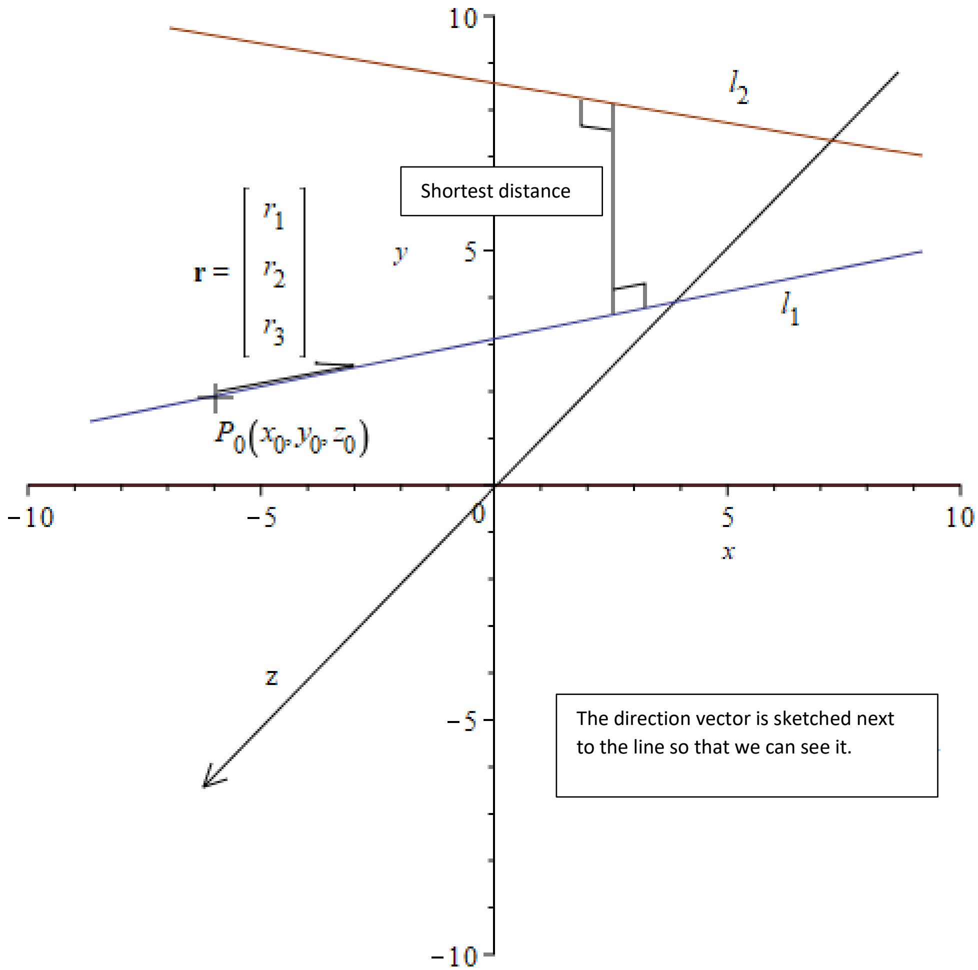
The distance from point (2, 0, 3) to plane x - 5y - 2z + 1 = 0 is

dist. = $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|1\cdot 2+(-5)\cdot 0+(-2)\cdot 3+1|}{\sqrt{2^2+0^2+3^2}} = \frac{3}{\sqrt{13}} \approx 0.832$

The straight line in space

We have seen five equations for the straight line: Two in ordinary mathematics and three in 2D vector mathematics, where the third is a vector function displaying a parametric curve.

When we look at the possibilities for an equation of the straight line in 3D, we can only use the vector function based on a point P_0 on the line and a known direction vector of the line. Just like in 2D, only now with the z coordinate added.



The vector function of the straight line deduces directly from line l_1 in the diagram. Starting from point P_0 the direction vector may move a point (the arrow head) up and down the line by multiplying with a parameter, we call t (parameter is Greek and here it means "along the measured"), and may be any real number:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \mathrm{t} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

which is the vector function (parametric function) of a straight line in 3D, where x_0, y_0, z_0 are coordinates of a point on the line, and t is the parameter. r_1, r_2, r_3 is a direction vector.

Lines may be parallel, intersecting, or as in most cases: skewed lines.

If there is more than just one line, the parameters must have different names, for instance t, s, etc. for each line.

The distance (shortest) between skewed lines

The diagram also shows line l_2 and the line between l_1 and l_2 with the shortest distance. Only one line fulfils this, and it is orthogonal with both l_1 and l_2.

Both lines have a known point we now call P_1 and P_2 (though not shown), and both lines have a known direction vector we call $\mathbf{r_1}$ and $\mathbf{r_2}$ (though not shown).

If we cross $\mathbf{r_1}$ and $\mathbf{r_2}$ we have a normal vector **n** which only can be placed on the distance line.

We now form a vector $\mathbf{P_1P_2}$ (not shown) and project it on to **n** using the projection-length-formula. This will render the shortest distance between the lines:

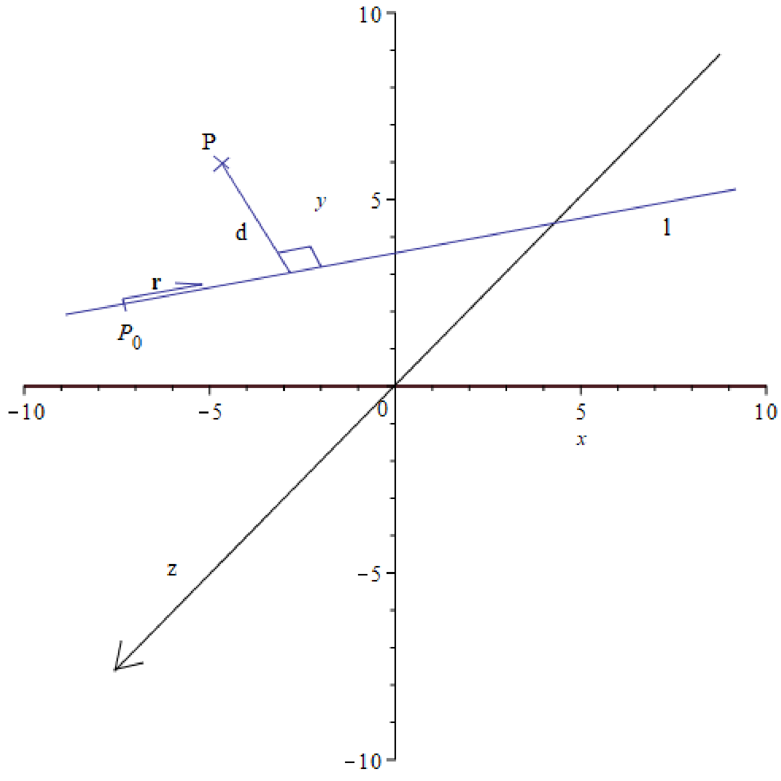
$|\mathbf{b_a}| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{a}|}$ in our case =>

$$\text{dist.}(l_1,l_2) = \frac{|\mathbf{n} \cdot \mathbf{P_1P_2}|}{|\mathbf{n}|}$$

where P_1 is a point on line l_1 and P_2 is a point on line l_2, while **n** is a common normal vector for the direction vectors of the lines.

The distance (shortest) point-line

The distance (d) from a point (P) to a line, is the same as the distance from the point to the direction vector of the line, thus from P to **r**.



P_0 is a known point on the line. We form a vector $\mathbf{P_0P}$ with the angle v between $\mathbf{P_0P}$ and l (not shown). Then

$$d = |\mathbf{P_0P}| \cdot \sin v$$

sin v is found from the formula we have shown in an example:

$$\sin v = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| \cdot |\mathbf{b}|} \qquad \text{here}$$

$$\sin v = \frac{|\mathbf{r} \times \mathbf{P_0P}|}{|\mathbf{r}| \cdot |\mathbf{P_0P}|} \qquad \Rightarrow \qquad \text{inserted in d}$$

$$d \;=\; |\,\mathbf{P_0P}\,| \cdot \frac{|\mathbf{r} \times \mathbf{P_0P}|}{|\mathbf{r}| \cdot |\mathbf{P_0P}|} \qquad \Leftrightarrow$$

$$d \;=\; \frac{|\mathbf{r} \times \mathbf{P_0P}|}{|\mathbf{r}|}$$

which is the distance formula point-line, where P is the point, **r** is the direction vector of the line, and P_0 is a point on the line.

The distance between two parallel planes

Planes are parallel if their normal vectors are proportional, for instance

$x - 5y - 2z + 1 \;=\; 0 \qquad$ and $\qquad -2x + 10y + 4z \;=\; 0$

where it is seen that

$$\begin{pmatrix} -2 \\ 10 \\ 4 \end{pmatrix} + (-2)\cdot\begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix} \qquad \text{the proportionality factor is } -2$$

Then, the distance is found by selecting a point in the one plane, and calculate the distance to the other plane using the distance formula point-plane.

The angle v between two planes

equals the angle between their normal vectors, which may be found in two ways, as shown earlier.

Either via the dot product

$$\cos v \;=\; \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|}$$

or via the cross product

$$\sin v = \frac{|\mathbf{n}_1 \times \mathbf{n}_2|}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|}$$

and then finish by the inverse function to isolate the angle (here called v).

The normal vector may be directed away from either side of the plane, so depending on the side chosen, we find either the acute angle (< 90°) or the obtuse angle (> 90°) between the planes.

The angle between line and plane

is found by the angle u between the direction vector of the line, and the normal vector of the plane. As above, it may be done either via the dot product

$$\cos u = \frac{\mathbf{r} \cdot \mathbf{n}}{|\mathbf{r}| \cdot |\mathbf{n}|}$$

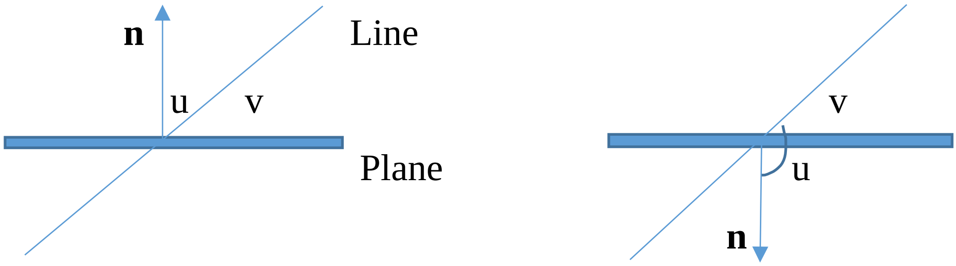
or via the cross product

$$\sin u = \frac{|\mathbf{r} \times \mathbf{n}|}{|\mathbf{r}| \cdot |\mathbf{n}|}$$

and go on by using the inverse function to isolate u. Finally, the angle v between line and plane is calculated by taking into account that **n** is turned 90° relative to the plane. Furthermore, since **n** may be on either side of the plane, our v must be found by

either $\quad v = 90° - u \quad$ or $\quad v = u - 90°$

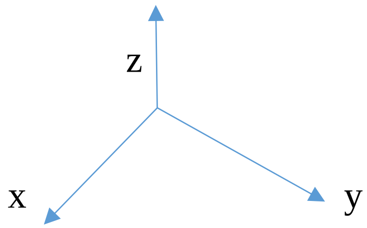
which is seen from the figure:



Examples

1.

We now chose a local coordinate system, where x and y are horizontal and z is vertical. This is often applied outdoors in the terrain. Surveyors utilize both global systems (gps) as well as local systems relative to known fixed points like the corner stone of an old building.



We will find the distance between two lines and calculate in meters:

The underside of a railway overpass follows a straight line through point (10, 10, 7) and has a direction vector $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ which renders this vector function

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad => \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

It is observed, that the direction vectors z coordinate is 0, thus the line (here the railway overpass) is horizontal.

The top of a motorway follows a straight line through point (5, 6, 2) and has a direction vector $\begin{pmatrix} -1 \\ 5 \\ 0.2 \end{pmatrix}$ which renders this vector function

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + s \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad => \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix} + s \begin{pmatrix} -1 \\ 5 \\ 0.2 \end{pmatrix}$$

It is observed that the direction vectors z coordinate is not 0, thus the line (here the motorway) is not horizontal.

It is observed, that the direction vectors are different, so the lines are skewed and will cross somewhere, where the shortest distance can be calculated from

$$\text{dist.}(l_1,l_2) = \frac{|\mathbf{n} \cdot \mathbf{P_1P_2}|}{|\mathbf{n}|}$$

We call the railway overpass l_1 and the motorway l_2.

P_1 is the known point on l_1: (10, 10, 7)

P_2 is the known point on l_2: (5, 6, 2)

Vector $\mathbf{P_1P_2}$ then is: $\begin{pmatrix} 5-10 \\ 6-10 \\ 2-7 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ -5 \end{pmatrix}$

The common normal vector is:

$$\mathbf{n} = \mathbf{r_1} \times \mathbf{r_2} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 5 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0,6 \\ -0.4 \\ 13 \end{pmatrix}$$

The numerator $\left| \begin{pmatrix} 0,6 \\ -0.4 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -4 \\ -5 \end{pmatrix} \right| = |-3 + 1.6 - 65| = 66.4$

the denominator $((0,6)^2 + (-0.4)^2 + 13^2)^{½} \approx 13.02$

combined dist.$(l_1,l_2) = \frac{66.4}{13.02} \approx 5.1$ meters.

2.

The centre line for a motorway has (as in example 1) the vector function

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix} + s \begin{pmatrix} -1 \\ 5 \\ 0.2 \end{pmatrix}$$

Relative to the same local coordinate system someone wishes to build a house with the coordinates (301, 411, 9) for the corner nearest the motorway. Let us calculate the distance between centre line and corner in meters:

$d = \frac{|\mathbf{r} \times \mathbf{PoP}|}{|\mathbf{r}|}$ where

numerator $|\ \mathbf{r} \times \mathbf{P_0P}\ | = \left| \begin{pmatrix} -1 \\ 5 \\ 0.2 \end{pmatrix} \times \begin{pmatrix} 301-5 \\ 411-6 \\ 9-2 \end{pmatrix} \right| =$

$$\left| \begin{pmatrix} -46 \\ 66.2 \\ -1885 \end{pmatrix} \right| = ((-46)^2 + 66.2^2 + (-1885)^2)^{½} \approx 1887$$

denominator $|\mathbf{r}| = ((-1)^2 + 5^2 + 0.2^2)^{½} \approx 5.103$

Combined $d = \frac{1887}{5.103} \approx 369.8$ meters

3.

We will find the distance between two parallel planes x - 5y - 2z + 1 = 0 and -2x + 10y +4z = 0

First we control that the planes are parallel, and find that their normal vectors are proportional (by a factor -2). So, they are parallel. (Otherwise it would make no sense to continue).

In the first plane we chose a point with x = 0 and y = 0 which we insert and get z = $\frac{1}{2}$

From the point (0, 0, $\frac{1}{2}$) to the other plane -2x + 10y + 4z = 0 the distance is

dist. = $\frac{|\,\mathrm{ax_1+by_1+cz_1+d}\,|}{\sqrt{a^2+b^2+c^2}}$ => here

dist. = $\frac{|\,(-2)\cdot 0+10\cdot 0+4\cdot 0{,}5+0\,|}{\sqrt{(-2)^2+10^2+4^2}} = \frac{|\,0+0+2+0\,|}{\sqrt{120}} = \frac{2}{\sqrt{120}} \approx$ 0.183

4.

Let us find the angle between the two planes in example 3. We know they are parallel, so the angle must be 0°. This time we control with the formula for the dot product

cos v = $\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1|\cdot|\mathbf{n}_2|}$ => here

numerator $\begin{pmatrix}1\\-5\\-2\end{pmatrix}\cdot\begin{pmatrix}-2\\10\\4\end{pmatrix}$ = -2 + (-50) + (-8) = -60

denominator $(1^2 + (-5)^2 + (-2)^2)^{½} \cdot ((-2)^2 + 10^2 + 4^2)^{½}$ = 60

combined cos v = $\frac{-60}{60}$ = -1 => v = 180°

Whether we get 0° or 180° depends on the direction of the normal vectors. Thus, an answer of 180° corresponds fine with parallel planes.

5.

A roof surface lies on a plane with the equation

$x + 0y - z = 0$ or $z = x + 0y$

The centreline of a chimney has the equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

In which point will the centre line of the chimney intersect the roof surface? (i.e. where does the line intersect the plane?).

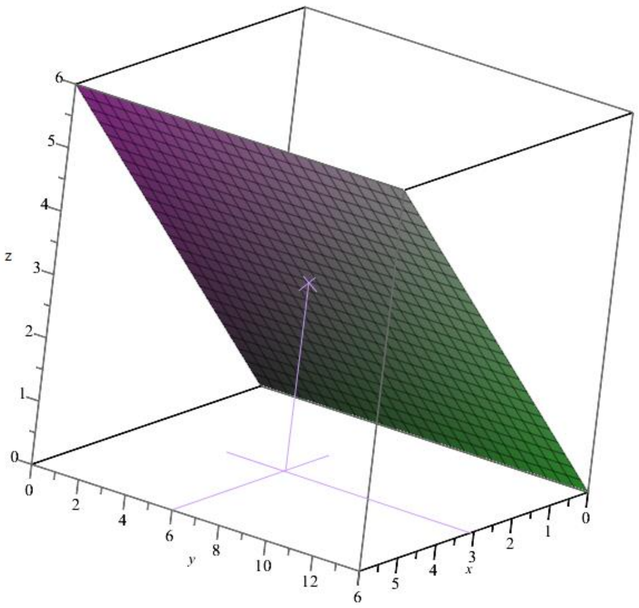
What is the acute angle between chimney and roof?

What is the roof angle relative to horizontal?

We need to include the y coordinate (even though it is 0) to show we are dealing with a plane. If we do not, one would reckon that we consider a line in 2D.

0y means that the plane is not crooked relative to the y-axis. y determines where we are in the length of the roof, while x and z determine the slope of the roof.

A plane is infinitely big, while the roof in the plane has a limited size. We will build a house 14 meters long and 12 meters wide. So we display a plot with x in the interval [0;6] and y in the interval [0;14]. z will become what the planes equation renders. We can solve the problem without a figure, but a plot is displayed in the diagram:



Please note that the divisions on the axis are not equidistant.

Answer:

The intersection point is where the equations of both the line and the plane have the same x, y, z - values. Thus, two equations with two unknowns.

We insert line into plane:

x + 0y - z = 0		=>
i.e.	3 + 0t	for x
and	nothing	for y

and $\quad 2 + 4t \quad$ for z

combined $\quad 3 - (2 + 4t) = 0 \quad \Leftrightarrow$

$1 - 4t = 0 \quad \Leftrightarrow$

$t = \frac{1}{4}$

so, when the ”running” parameter t is $\frac{1}{4}$, we are at the intersection. This t is inserted in the line equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix} + \left(\frac{1}{4}\right) \cdot \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

Which yields the intersection point (x, y, z) = (3, 6, 3) which also is shown in the figure with some helping lines.

The angle between chimney and roof:

$\cos u = \frac{\mathbf{r} \cdot \mathbf{n}}{|\mathbf{r}| \cdot |\mathbf{n}|} \quad \Rightarrow \quad$ here

numerator $\quad \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -4$

denominator $\quad (4^2)^{½} \cdot (1^2 + (-1)^2)^{½} = 4\sqrt{2}$

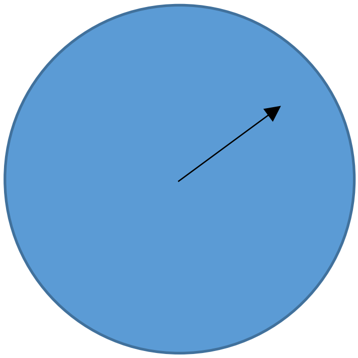
combined $\quad \cos u = \frac{-4}{4\sqrt{2}} \quad \Rightarrow \quad u = 135°$

Due to the direction of the normal vector, we found the obtuse angle. The acute angle between chimney and roof is 180° - 135° = 45°.

The angle between roof and horizontal is

vertical - 45° = 90° - 45° = 45°

The sphere



The arrow shows a radius from the centre C to a point P on the spherical shell.

C has the coordinates (a, b, c) and P has the coordinates (x, y, z) in a 3D coordinate system.

The distance between C and P is the radius, r.

The distance between two points in 3D was earlier derived as

$|P_1P_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ Pythagoras in 3D

which here is

$r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$ the equation of the sphere

Tangent plane

The equation for a tangent plane in a point on the sphere is determined by forming a normal vector from the centre to the point and use it in the equation of the plane.

Example 1

Sphere: $3^2 = (x - 0)^2 + (y - 1)^2 + (z - 2)^2$

which has the centre in C(0,1,2) and radius $r = 3$

Find the equation of the tangent plane in point P(0,1,5) $= (x_0, y_0, z_0)$ in the plane

We form the normal vector of the plane end minus start

n = **CP** = (0 - 0 , 1 - 1 , 5 - 2) = (0,0,3)

inserted into the planes equation

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ =>

$0(x - 0) + 0(y - 1) + 3(z - 5) = 0$ ⇔

$3z - 15 = 0$ ⇔

$z = 5$ Which is the equation of the tangent plane.

Part 5. Statistics

Very unusually, statistics is imprecise mathematics. Back in the day, it was discussed if the subject should - or should not - become a field within mathematics. The imprecise argues against, but the use of figures and calculations argues pro. So, it was decided that statistics is a field within mathematics.

Statistics is necessary when we handle big amounts of data, which in the real world will imply exceptions and shortcomings, making it impossible to be precise.

Therefore, we need concepts like average value, mean value, dispersion, standard deviation and some tool to estimate the precision of our calculations. A good question remains to be: How many observations/information/data do we need to make an estimation at all?

This we will discuss in a brief way, and we start by considering non-grouped observations.

Observations

Non-grouped observations

A candy bag contains 30 sweets. They weigh almost the same, but not quite. The information is overviewed from the table:

Observation. Here weight per sweet in grams	Number of observations	Frequency, f Out of 1	Frequency Out of 100, i.e. in %	Cumulative frequency Frequencies summarized
2.1	1	0.0333	3.3	0.0333
2.2	5	0.167	16.7	0.2
2.3	7	0.233	23.3	0.433
2.4	7	0.233	23.3	0.666
2.5	6	0.200	20.0	0.866
2.6	4	0.133	13.3	1
Total:	*30*	*1*	*100 %*	*1*

Observations and number of observations are measured, while frequency and cumulative frequency are calculated.

The manufacturer tries to achieve an average weight of 2.35 grams per sweet. We also observe, that 2.35 is the average of the values from 2.1 to 2.6 grams:

$$\text{average} = \frac{2.1+2.2+2.3+2.4+2.5+2.6}{6} = 2.35 \text{ gram}$$

However, the sweets are not equally dispersed between heavier or lighter sweets. So, we have to be more precise and calculate the mean value. The mean value is the mean of all observations and calculated by saying: observation multiplied by frequency, plus the next, etc. - all of it divided by the total number of sweets:

$$\text{mean value} = \frac{2.1\cdot 1+2.2\cdot 5+2.3\cdot 7+2.4\cdot 7+2.5\cdot 6+2.6\cdot 4}{30} = 2.38 \text{ grams}$$

Or by saying: observation times frequency plus the next, etc.:

mean value =

2.1·0.333+2.2·0.167+2.3·0.233+2.4·0.233+2.5·0.2+2.6·0.133 =

2.38 grams

So, the consumer gets a little more for the money.

Quartiles, is the division of the number of data points into four parts, or quarters. The first quartile is equal to - or bigger than - 25 % of the data, which may be read in the column "Cumulative frequency". The second quartile (also called the median) is ≥ 50 % of the data. The third quartile is ≥ 75 % of the data, and finally, the fourth quartile comprises all data, 100 %, of the whole series of observation. Quartiles may also be called percentiles.

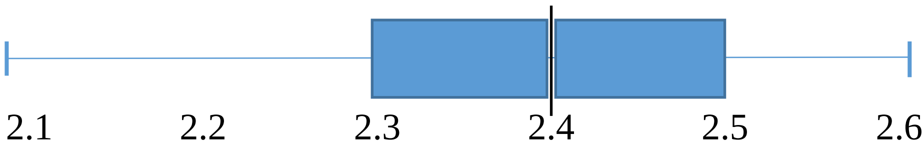
A set of quartiles consists of the first, the second, and the third quartile. For our sweets it is: [2.3 ; 2.4 ; 2.5].

We get a good overview by making a box chart:

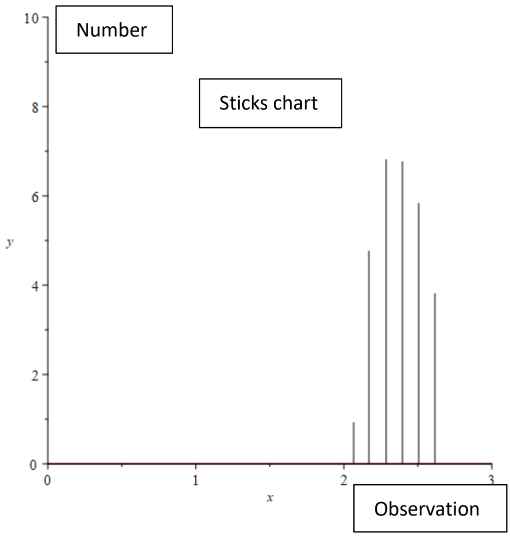
smallest observation - 1.quartile - 2.quartile - 3.quartile - biggest observation

Here: 2.1 - 2.3 - 2.4 - 2.5 - 2.6

and the box chart:



Or a sticks chart:



Grouped observations

In a ball bearing production are produced spheres with a diameter of 5.00 millimeters. Some are a little bigger, and some are a little smaller. This is compensated for by using small spheres with thick rings, and the bigger spheres are used with the thinner rings. This way nothing is wasted, and the ball bearings will have the dimension required. We gather the spheres in groups as follows:

Observation. Here diameter	Number of observations	Frequency, f Out of 1	Frequency Out of 100 i.e. in %	Cumulative frequency Frequencies summarized
[4.80 ; 4.85]	1	0.01	1	0.01
]4.85 ; 4.90]	5	0.05	5	0.06
]4.90 ; 4.95]	21	0.21	21	0.27
]4.95 ; 5.00]	33	0.33	33	0.6
]5.00 ; 5.05]	31	0.31	31	0.91
]5.05 ; 5.10]	9	0.09	9	1
Total	*100*	*1*	*100 %*	*1*

When calculating the mean value, we use the diameter in the middle of the interval, i.e. 4.825 mm, 4.875 mm, etc. Thus, we calculate somewhat imprecise, but that is accepted:

mean value =

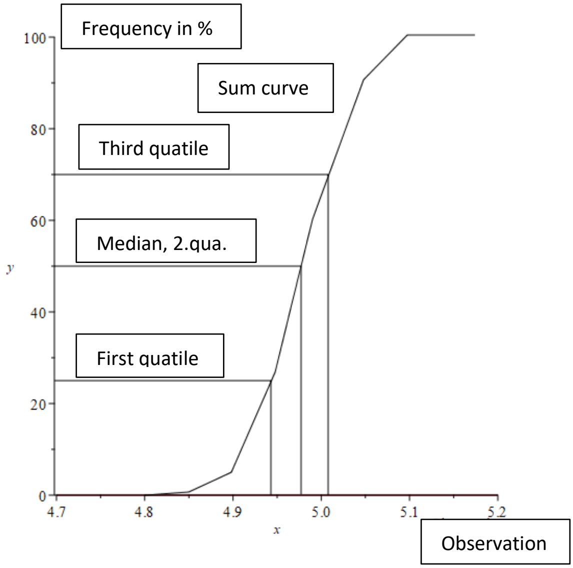
$4.825 \cdot 0.01 + 4.875 \cdot 0.05 + 4.925 \cdot 0.21 + 4.975 \cdot 0.33 + 5.025 \cdot 0.31 + 5.075 \cdot 0.09 =$

4.983 mm.

The production plant is working well though it may be fine adjusted.

It would be too coarse to read the set of quartiles from the table.

It is possible to sketch a box chart or a sticks chart, but it is better to sketch a sum curve in a diagram:



On the first axis are the observations from the table.

On the second axis are the cumulative frequencies from the table.

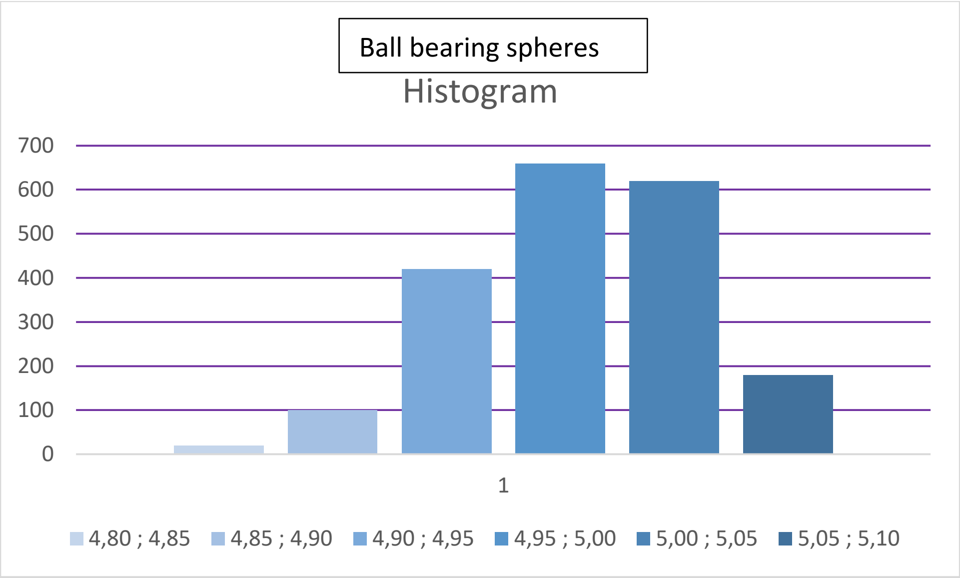
Now we can have a finer reading of the set of quartiles from the sum curve: We go from 25%, 50%, 75% horizontally to the curve and vertically to the first axis to read the set of quartiles: [1.qua. ; 2.qua. ; 3.qua.] = [4.94 ; 4.976 ; 5.01]

Since it is a reading, it will be inaccurate (with some uncertainty).

The more data we have, and the more intervals we define - the finer and smoother the sum curve will be.

Finally, we will show the histogram, which is a special column diagram displaying the history of the measurements, such that the area of each column corresponds with the frequency of the interval.

Here we display the production of ball bearings with data from the table:



The groups are on the first axis. On the second axis is the number we multiply by the width of the observation to render the frequency in per cent, %. Here the width of all the groups is 0.05:

(4.85 - 4.80) · 20 = 1 %   0.05 · 100 = 5 %   0.05 · 420 = 21 %

0.05 · 660 = 33 %   0.05 · 620 = 31 %   0.05 · 180 = 9 %

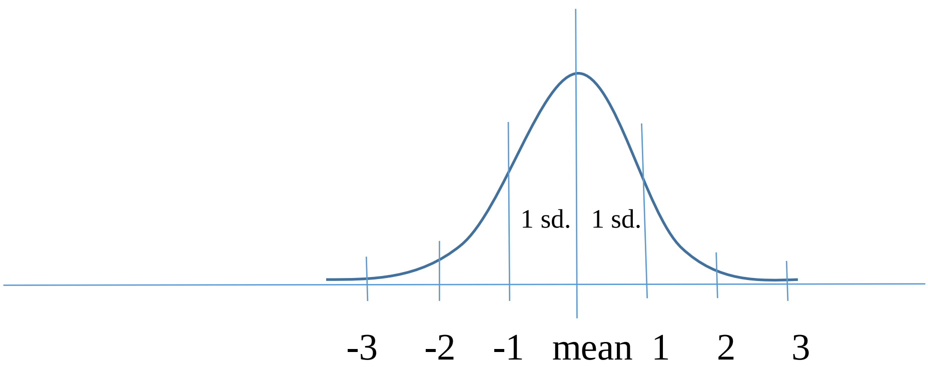
We find that the total is 100%, just as it should be.

Thus, the second axis of the diagram does not show something directly useful. It is the area, that shows the frequency. In a way, the histogram is the precursor of the normal distribution curve (see below) which also has the area 1 or 100% under the curve. Otherwise, the advantage of this diagram is mainly for measurements with different widths.

Altogether, it is fair to say that sticks charts and sum curves often gives the best overview.

Normal distribution, variance and standard deviation

Many observations are normal distributed, meaning that for instance the diameters of the spheres will disperse symmetrically around the mean value of 5 mm. This was not exactly so in our case, but if we were observing 1000 or 10 000 spheres, they would probably be normal distributed. A normal distribution curve may look like this:



Area: 0,1% 2,3% 13,6% 34,1% 34,1% 13,6% 2,3% 0,1%

If we sketch a sticks diagram or a histogram from many normal distributed measurements, we will have a bell like shaped curve as shown.

The figure may be higher or lower depending on the division of the scale. The point is that all observations are under the curve, so that the area under the curve includes all (100 %) observations of for instance the spheres of our ball bearings. Thus, the area under the curve is 1 or 100 %.

There are shown three lines on either side of the mean value. The distance between two lines is called the standard deviation, sd, (or dispersion). There are three standard deviations to the left, and three standard deviations to the right.

the area between line -1 and 1 include 68.2 % of the measurements

the area between line -2 and 2 include 95.4 % of the measurements

the area between line -3 and 3 include 98.8 % of the measurements

The last two times 0,1 % are further away from the mean value, where the normal distribution curve approaches the first axis asymptotically.

In science we usually present data this way:

mean value $\pm$ standard deviation

Thus 68.2 % of the measured data lies within ± the standard deviation (i.e. between line -1 and 1). Then we know what the whole distribution curve looks like.

Yet how do we calculate the standard deviation?

First, it should be decided what should be called the standard deviation.

The thinking was to consider how far an observation is from the mean value, and then magnify the distance by squaring, and at the same time having a positive number, whether we are below or above the mean value:

(obs. - mean)^2

and multiply by the significance, i.e. the frequency, for each observation

f · (obs. - mean)^2

and summarize for all the observations

Σ f · (obs. - mean)^2

Σ means ”the sum of”. A full mathematical expression

$$\sum_{i=1}^{n} f_i \cdot (x_i - \mu)^2$$

Where the mean value is called μ (the Greek letter ”my”), the observation is called x_i , i (for integer) is the number of the observation from 1 (start) to n (end).

Thus, we define the variation as

$$\text{Var.} = \sum_{i=1}^{n} f_i \cdot (x_i - \mu)^2$$

which expresses how far away we are from the mean value.

Only, it turned out to be a little too far away, so we continue by taking the square root to get the standard deviation

$$\text{standard deviation} = \sqrt{Var.}$$

The standard deviation is often called σ (a Greek sigma)

$\sigma = \sqrt{Var.}$

Thus, we can now calculate the standard deviation from the variance.

Scientists have agreed on presenting data as

mean value ± standard deviation ⇔ $\mu \pm \sigma$

Example 1

The mean value for the spheres of the ball bearing was previously calculated as

$\mu = 4.983$ mm.

Even though we do not have so much data, and even though the distribution is uneven with more spheres bigger than mean and fewer smaller than mean, we assume, that if we had much more data, we would have a normal distribution. Subject to this condition the standard deviation may be calculated as

$$Var. = \sum_{i=1}^{n} f_i \cdot (x_i - \mu)^2 \quad =>$$

$$Var. = 0.01\cdot(4.825 - 4.983)^2 + 0.05\cdot(4.875 - 4.983)^2 + 0.21\cdot(4.925 - 4.983)^2 +$$
$$0.33\cdot(4.975 - 4.983)^2 + 0,31\cdot(5.025 - 4.983)^2 + 0.09\cdot(5.075 - 4.983)^2 =$$
$$0.00282 \quad =>$$

$$\sigma = \sqrt{Var.} = \sqrt{0.00282} = 0.0531 \text{ mm.}$$

Combined the mean value and the standard deviation are:

$\mu \pm \sigma$ => 4.983 ± 0.053 mm.

Not all measurement series are normal distributed. Then we will have a skewed distribution curve. A further treatment of this falls outside the scope of this book.

Chi to the power of two - testing

We may form a quotient to evaluate if some observations (intervals or whole series) are close to - or further away from - what was expected:

$$\chi^2 = \Sigma \frac{(observation - expected\ observation)^2}{expected\ observation}$$

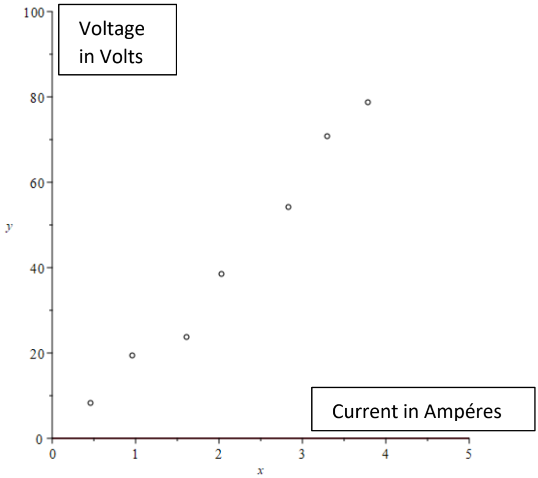
we use the ancient Greek letter χ = Chi and the quotient is called a chi to the power of two - test. In the numerator, we magnify the deviation by squaring, and at the same time get a positive number no matter if we are above or below the expected, - same way of thinking as for the variance.

If the observed equals the expected, the quotient is zero, which is perfect. A bigger distance between observed and expected renders a bigger quotient.

Regression

In mathematics, we use the concept: Regression, to organize measurements in a best-fit function.

We have been to the laboratory to measure currents (I) and voltages (U) in a circuit. We measured this:



We know that the measurements should follow Ohm´s law

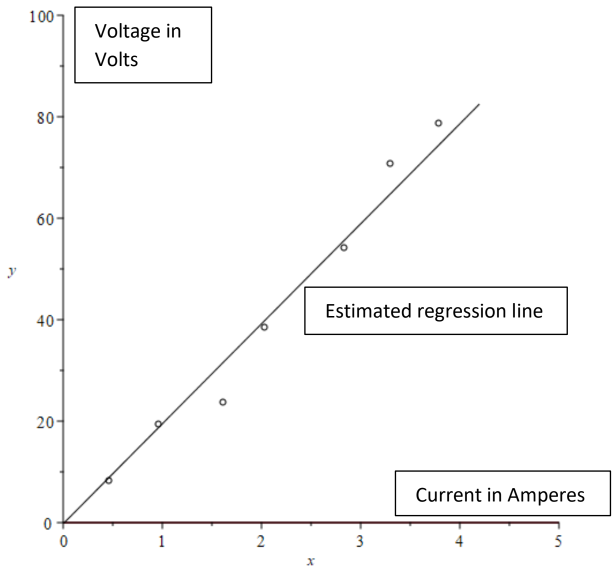
$U = R \cdot I$ where R is the resistance in Ohms.

This corresponds with the equation for a straight line through Origo (0,0):

$y = a \cdot x$

Now the second axis is U, the first axis is I, and the slope is R. Thus, Ohm’s law displays a straight line in a I,U diagram, and thus we know that our measurements should form a straight line. However, there are uncertainties in our measurements, so which straight line is the best fit?

We may right away place a ruler to see, that this line is nearly perfect:



Things become much more complicated if we want to determine the best-fit line by calculation. We use the "method of least squares" in which we guess a line, find the vertical distance between a point of measurement and the line, and square it (same way of thinking as for the variance). We do so for all measured points and summarize the squares. Then we guess other lines, calculate new squares, and choose the line with the least square. This line is our calculated regression line - or "best-fit curve".

Clearly, this is a major calculation work suitable for advanced CAS.

Regardless of using a calculator or a calculation program, the principle is the same. In our example:

1. We enter the measured data for the first axis and call them I
2. We enter the measured data for the second axis and call them U

3. We ask for linear regression using $y = ax + b$ or we write a key line such as: *LinReg(I,U)*
4. Enter

Then, we will have a diagram with a regression line and its equation.

Often we also get a reliability factor - often called r or R (maybe squared: r^2 or R^2) - informing how reliable the program finds the calculation. 1 is very fine, 0,99 is fine; 0,95 is not as fine, etc.

The principle is the same for other functions:

For example, Time on the first axis and Number on the second axis for the regression of an exponential function $y = b \cdot a^x$ with the key line: *ExpReg(Time, Number)*

Or Time on the first axis and Pounds on the second axis for the regression of a power function $y = b \cdot x^a$ with the key line: *PowReg(Time, Pounds).*

Probability and combinations

Statistics is when we process a large number of data. One can argue that we look to the past.

Probability is when we seek a probable answer to something happening in the future.

Simple probabilities can be calculated precisely, for instance the probability of observing six when we roll a dice. No uncertainty.

Complicated probabilities based on many data, all of which holds their own uncertainty, will lead to an uncertain probability. For example a weather forecast on having sunshine tomorrow at 2 a.m. In such a case we may calculate/estimate a probability expressed by a number between 0 and 1 or between 0 and 100% and an uncertainty. For instance a probability $81\% \pm 5\%$ of having sunshine tomorrow at 2 a.m. Usually the public will only be informed about the 81%.

Complicated probability calculus is a large area of expertise for specialists, such as meteorologists and insurance professionals.

Here we will deal with simple and precise probability calculation, which nevertheless quickly becomes quite difficult. We will see some formulas for Permutation and Combination that are so difficult to prove that we will instead show their correctness by examples - and this is done in the chapter: "Rarely used proofs and calculations".
It should also be mentioned that there are many technical terms in this subject.

We will use new numbers such as

5! which stands for 5 factorial, and which means $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ and which, by the way, gives 120.

7! which stands for 7 factorial, and which means $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ and which, by the way, gives 5040.

And a little specialty: $0! = 1$ which we have to define in order to avoid having zero erasing everything in the formulas we are going to use.

Theory

The probability is a number between 0 and 1 (or between 0 and 100%) calculated as

$$probability = \frac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes}$$

So we need to know both the numerator and the denominator in order to calculate the probability.

First we find the denominator *number of possible outcomes* which we determine according to the following six cases where n is the number of elements/objects in the pool/population and r is the number considered:

1. *"Both, and"* with the technical term Multiplication rule:

   number of possibilities = one possibility times the other

2. *"Either, or"* with the technical term Addition rule:

   number of possibilities = one possibility plus the other

3. The formula for *any order, without repetition* (Technical term: Permutation ≈ reversal of order)

   $$number\ of\ possibilities = P = \frac{n!}{(n-r)!}$$

   Will be shown in chapter "Rarely used proofs and calculations"

4. The formula for *no order, without repetition* (technical term: Combination)

$$number\ of\ possibilities\ = K = \frac{n!}{r!\cdot(n-r)!}$$

Will be shown in chapter "Rarely used proofs and calculations"

5. *Any order, with repetition* agrees with point 1: *"Both and"*, therefore:

$$number\ of\ possibilities\ = n^r$$

6. The formula for *no order, with repetition*

$$number\ of\ possibilities\ = \frac{(n-1+r)!}{r!\cdot(n-1)!}$$ will not be shown

This formula is also - and especially - used in connection with random sampling, which we will come back to.

These six cases/formulas show how we combine the possibilities – and the technical term is Combination.

In the following examples we first calculate the number of possible outcomes (the denominator) – then the probability (the whole fraction):

Examples

1.

How many combinations are there if we roll a dice two times?

The case is *"Both, and"* which renders: $6 \cdot 6 = 36$ possibilities

----

What is the possibility of having two sixes in a row?

$$probability\ first\ roll = \frac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes} = \frac{1}{6}$$

$$Probability\ second\ roll = \frac{1}{6}$$

The case is *"Both, and"* which renders: $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} \approx 2.78\%$

2.

How many "eyes" are there on two dices?

Each dice has six sides with each 1, 2, 3, 4, 5, 6 "eyes".

The case is *"Either, or"* which renders: 6 + 6 = 12 possibilities

----

What is the possibility of having one six in two rolls with one dice?

$$probability\ first\ roll = \frac{1}{6}$$

$$probability\ second\ roll = \frac{1}{6}$$

The case is *"Either, or"* which renders: $\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \approx 33.3\%$

3.

A foreman and deputy foreman and alternate must be elected in a board with 7 members. The one first elected becomes foreman, the next becomes deputy foreman, and the third becomes alternate. In how many ways can the 3 people be elected?

Here the order is crucial, and the first elected will not be put back in the pool. Therefore the case is *any order, without repetition.* The formula is:

$$P = \frac{n!}{(n-r)!}$$

Where P is the number of possibilities, n is the number of members (here 7), and r is the number of selected people (here 3).

The values are inserted:

$$P = \frac{n!}{(n-r)!} = \frac{7!}{(7-3)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{1} = 210 \; possibilities$$

What is the probability of Liz becoming foreman, Peter becoming deputy foreman, and Ann becoming alternate?

Liz and Peter and Ann is one possibility (the favorable outcome), therefore:

$$probability = \frac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes} = \frac{1}{210} \approx 0.476\%$$

4.

A foreman and deputy foreman and alternate must be elected in a board with 7 members. The election will show who of the 3 people are having the 3 positions – regardless of which position. The decision amongst the 3 is postponed until later. How many possibilities are there for selection of the 3 people?

Here the order does not matter, and the one selected will not be put back into the pool, therefore: *no order, without repetition*. The formula is:

$$K = \frac{n!}{r! \cdot (n-r)!}$$

Where K is the number of possibilities, n is the number of members (here 7), and r is the number selected (here 3). The values are inserted:

$$K = \frac{n!}{r!(n-r)!} = \frac{7!}{3!(7-3)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 \cdot 4} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35 \; possibilities$$

What is the possibility of the election of Liz and Peter and Ann?

Liz/Peter/Ann or other orders amongst them is one possibility, therefore:

$$probability = \frac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes} = \frac{1}{35} \approx 2.86\%$$

5.

A code is formed from three small letters from an alphabet with 25 letters – and three digits. How many possibilities are there?

The case is *any order, with repetition* => *"Both and"* => multiplication:

$25 \cdot 25 \cdot 25 \cdot 10 \cdot 10 \cdot 10 = 25^3 \cdot 10^3 =$ 15 625 000 possibilities

----

What is the probability of having the code abc123?

abc123 is the only favorable outcome, therefore:

$$probability = \frac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes} = \frac{1}{15\ 625\ 000}$$

The probability of having *either* the code abc123 *or* the code bcd123 is:

$$probability = \frac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes} = \frac{1+1}{15\ 625\ 000} = \frac{2}{15\ 625\ 000}$$

6.

A driving school must have four new cars. The dealer has seven models that can be used. How many possibilities are there to choose cars if the same combination has may be repeated?

The case is *no order, with repetition:*

$$\frac{(n-1+r)!}{r! \cdot (n-1)!} = \frac{(7-1+4)!}{4! \cdot (7-1)!} = \frac{10!}{4! \cdot 6!} = 210\ possibilities$$

Three models are from last year, which the seller hopes he can sell together with one of the new cars. What is the probability of that?

We name the cars A1, A2, A3, B, C, D, E.

A1, A2, A3, B can be combined in 4 ways.

A1, A2, A3, C can be combined in 4 ways.

A1, A2, A3, D can be combined in 4 ways.

A1, A2, A3, can be combined in 4 ways.

The case for the numerators number of favorable outcomes is *”Either, or”* => addition. Therefore:

$$probability = \frac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes} = \frac{4+4+4+4}{210} = \frac{16}{210} \approx 7.6\%$$

And now some more advanced combinations:

7.

A country's three states come together to hold a football tournament. 10 matches are to be held, distributed so that the largest state A must hold 5 matches, the second state B must hold 3 matches, and state C gets 2 matches - but which ones?
5 A notes, 3 B notes and 2 C notes are put in a jar. A total of 10 notes.
Three notes are drawn for the first three matches. What is the probability that they are all A?

The order is irrelevant so the situation is *no order, without repetition.*

We must calculate $probability = \dfrac{number\ of\ favorable\ outcomes}{number\ of\ possible\ outcomes}$

The numerator shows how many possibilities there are of having 3 out of 5 A´s. =>

Numerator $K = \frac{n!}{r!\cdot(n-r)!} = \frac{5!}{3!\cdot(5-3)!} = \frac{120}{6\cdot 2} = 10\ favorable\ outcomes$

Taken from the whole pool/population which is the denominator:

Denominator $K = \frac{n!}{r!\cdot(n-r)!} = \frac{10!}{3!\cdot(10-3)!} = \frac{7!\cdot 8\cdot 9\cdot 10}{6\cdot 7!} = 120\ possibilities$

Combined $probability = \frac{favorable\ outcomes}{possible\ outcomes} = \frac{10}{120} = \frac{1}{12} \approx 8.33\%$

8.

The three notes showed A, B, B and are not put back. Two new notes are drawn. What is the possibility of B, C ?

The order is irrelevant so the case is *no order, without repetition.*

Left are now A, A, A, A, B, C, C. i.e. 7. =>

Denominator $K = \frac{n!}{r!\cdot(n-r)!} = \frac{7!}{2!\cdot(7-2)!} = \frac{5!\cdot 6\cdot 7}{2\cdot 5!} = 21\ possibilities$

Numerator for B $K_B = \frac{1!}{1!\cdot(1-1)!} = \frac{1}{1\cdot 1} = 1$

Numerator for C $K_C = \frac{2!}{1!\cdot(2-1)!} = \frac{2}{1\cdot 1} = 2$

K_B and K_C must be combined as *"Both, and"* => multiplication =>

Combined $probability = \frac{favorable\ outcomes}{possible\ outcomes} = \frac{K_B\cdot K_C}{K} = \frac{1\cdot 2}{21} = \frac{2}{21} \approx 9.5\%$

Binomial distribution, random sample, and confidence interval

Binomial distribution, introduction

In probability theory, a random variable is named a *stochastic variable*. The name comes from ancient Greek.
A binomial experiment with a random variable has three properties:

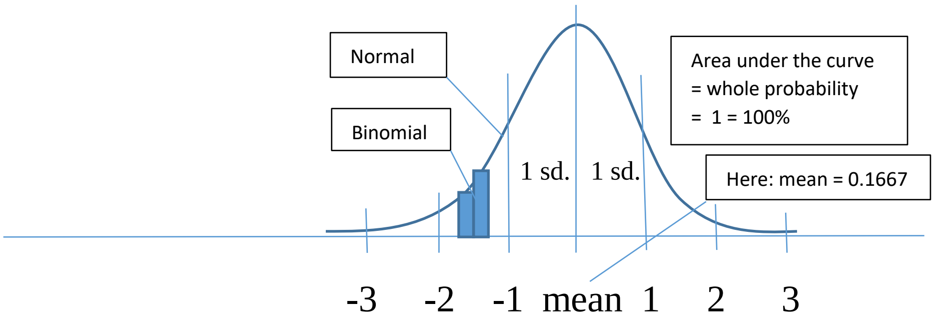
1. The outcome is a success or a failure (hence the name binomial).
2. Experiment 1 has no influence on Experiment 2, which has no influence on Experiment 3, etc. – all experiments are independent.
3. All trials have the same probability of success.

Good examples are hitting flats/crowns with a coin - or rolling with a dice.

Rolling a dice

When rolling a dice there is $\frac{1}{6}$ (≈ 0.1667) probability of having 1, $\frac{1}{6}$ probability of having 2, etc., in an infinite number of rolls.

With an finite number of rolls we can sketch a diagram with r (number) on the first axis, and P (probability) on the second axis. Then we obtain a *binomial distribution*, which equals the *normal distribution* in Statistics with mean value and standard deviation:



Area:	<0,1%	2,3%	13,6%	34,1%	34,1%	13,6%	2,3%	<0,1%

We call it the normal distribution curve, when it describes the observation, - for instance the diameter of a sphere.

We call it the binomial distribution curve, when it describes the probability of the number of successes.

A detail: A normal distribution curve is smooth (continuous) because the measurement quantity (for example a sphere diameter) can be any value – with a smooth transition. A binomial distribution curve will be discontinuous because the number of success/failure is a number without a smooth transition to the next number of success/failure. One also uses the technical expressions that the normal distribution curve is continuous, while the binomial distribution curve is discrete (= separated). However, if the binomial distribution curve has a lot of points (possibly shown as bars) it will in practice be smooth.

If we roll only 24 times, it is not for certain, that we will see 4 of each (4 times six, four times five, etc.).

One may call the 24 rolls a *random sample*, which makes it obvious, that we need a tool for the assessment of the random sample.

First, however, we must consider more theory and formulas for the binomial distribution:

The binomial distribution

Thus, a binomial distribution curve is ”similar” to the normal distribution curve. And the whole area under the curve shows a probability of 1 or 100%.

However, since we now use the curve to describe the probability, we use other data/information than within statistics, wherefore we need to change the calculation method of finding the mean value and the standard deviation, as well as finding a formula (the function) for the binomial distribution curve.

The function must describe a *no order* sequence of *succes* and *failure*. The three sizes must be combined *"Both, and"* => multiplication:

Binomial distribution (P) = no order · succes · failure

Here P stands for probability – and *not* for Permutation as before.

The combination for *no order* is $K = \frac{n!}{r! \cdot (n-r)!}$

Of the whole population/pool with the number n there are r with a probability of succes p (a number between 0 and 1). More successes must be multiplied i.e. p^r

Those which are not successes must be failures with the number (1-p). More failures must be multiplied i.e. $(1-p)^{n-r}$

Combined the binomial function is:

$$P = \frac{n!}{r! \cdot (n-r)!} \cdot p^r \cdot (1-p)^{n-r}$$

The most probable value is the mean value which is determined as

mean value = number times probability =>

$$\mu = n \cdot p$$

The standard deviation for the size x is known from Statistics as

$$\sigma(x) = \sqrt{\sum_{i=1}^{n} f_i \cdot (x_i - \mu)^2}$$

which by a long and difficult conversion becomes:

$$\sigma(x) = \sqrt{n \cdot p \cdot (1-p)}$$ or just σ since the variable is not always called x

we will not show the conversion

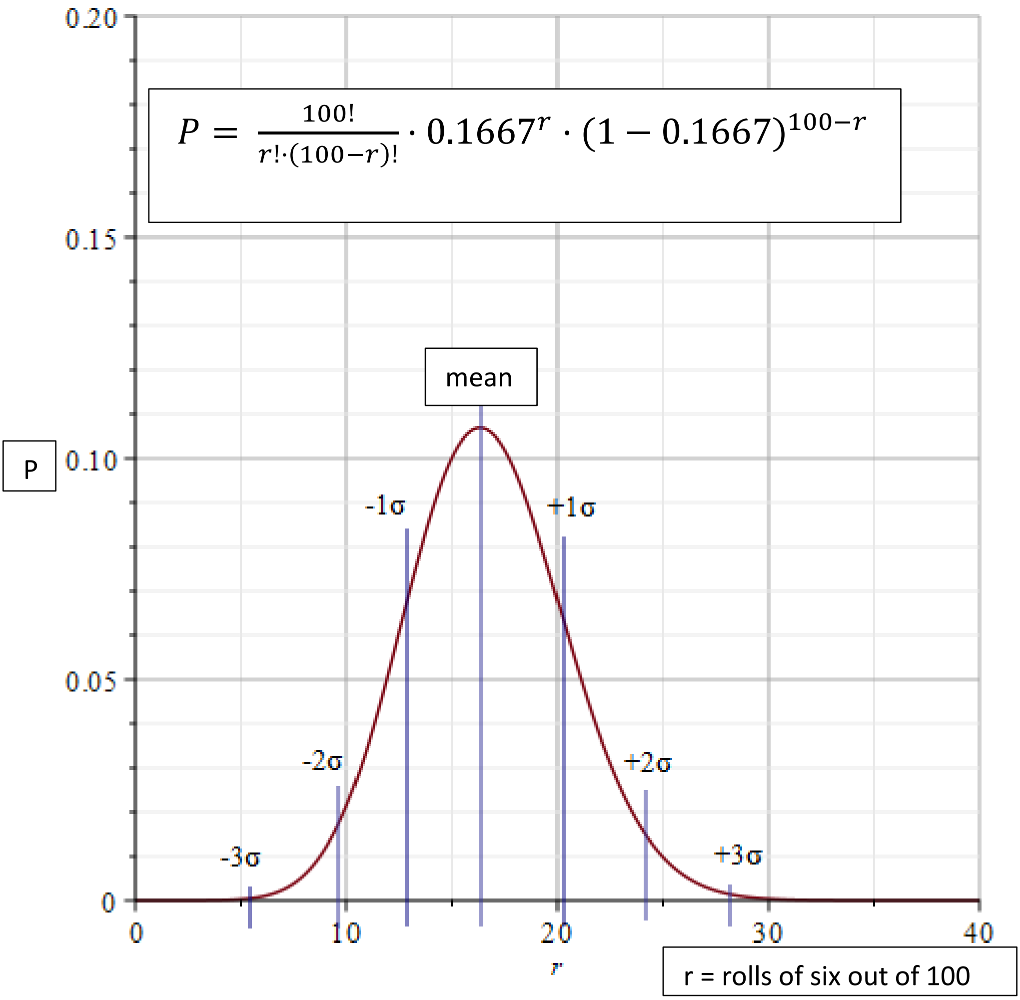
i.e. now with sizes for binomial probability calculation.

Which, by the way, gives the variance $Var = n \cdot p \cdot (1 - p)$

Rolling a dice

We roll a dice n =100 times. Each roll, for instance having a 6, has a probability of p = $\frac{1}{6}$ ≈ 0.1667

Then the binomial distribution (number-probability diagram = r-P diagram) looks this way:



Area ca.;	<0.1%	2.3%	13.6%	34.1%	34.1%	13.6%	2.3%	<0.1%	= 100%

the area between line - 1σ and + 1σ is 68,2 % of the measurements.

the areal between line - 2σ and + 2σ is 95,4 % of the measurements.

the areal between line - 3σ and + 3σ is almost 100 % of the measurements.

The mean value (= the most probable number of successes) is:

$mean = \mu = n \cdot p = 100 \cdot 0.1667 = 16.67$ rolls with six eyes

The standard deviation is:

$$\sigma = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{100 \cdot 0.1667 \cdot (1-0.1667)} = 3.727$$

Both values may be read in the diagram.

If we instead roll the dice 48 times, we expect to have:

$\mu = n \cdot p = 48 \cdot 0.1667 = 8$ rolls with 6 eyes

but the standard deviation is big:

$\sigma = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{48 \cdot 0.1667 \cdot (1-0.1667)} = 2.58$

We see that the standard deviation is relatively smaller the more rolls we do. In other words, the more times we roll the dice, the more certain we become to have $\frac{1}{6}$ sixes.

Random sample and confidence interval for a binomial distribution

Rolling a dice is a simple case, since we just have to consider 6 possibilities. We know that there always is $\frac{1}{6}$ probability of having six in the next roll.

Otherwise uncertain is it with a big amount of data. Then we will have to take *random samples* as representative as possible, followed by an estimation/calculation of the confidence in this random sample. We do so by calculating the *confidence interval,* which is a ”statistic uncertainty of the probability”:

For the random sample we can calculate the mean value and the standard deviation, so that we can sketch the binomial curve of the random sample.

As mentioned the mean value is $\mu = n \cdot p \quad \Leftrightarrow \quad p = \frac{\mu}{n}$

We have seen the standard deviation of all the sizes x:

$$\sigma(x) = \sqrt{n \cdot p \cdot (1-p)}$$

Now we see it as the standard deviation of a single size =>

$$\sigma\left(\frac{x}{n}\right) = \frac{\sqrt{n \cdot p \cdot (1-p)}}{n} = \sqrt{\frac{n \cdot p \cdot (1-p)}{n^2}} = \sqrt{\frac{p \cdot (1-p)}{n}}$$

and if we choose the mean value as the single size, we have

$$\frac{x}{n} = \frac{\mu}{n} = p$$ =>

$$\sigma(p) = \sqrt{\frac{p \cdot (1-p)}{n}}$$

which is the standard deviation σ as a function of the point-probability p and the number of all the sizes n

In a random sample we usually choose the mean value as the size in focus, which for the random sample is called $\mu*$ or the point-estimate $p*$ (some formula tables call it p with a "hat")

The confidence interval is often chosen to be from -2 to +2 (see the constant) spreads, i.e. 95,4% - often rounded as 95% =>

$$Confidenceinterval = \left[p^* - 2\sqrt{\frac{p^* \cdot (1-p^*)}{n}} \;\; ; \;\; p^* + 2\sqrt{\frac{p^* \cdot (1-p^*)}{n}} \right]$$

The confidence interval here shows, that there is a 95,4% probability, that the actual mean value is found within the interval.

One may also choose another confidence interval, for instance 68,2% within ± 1σ

$$Confidence\ interval = \left[p^* - \sqrt{\frac{p^* \cdot (1-p^*)}{n}} \;\; ; \;\; p^* + \sqrt{\frac{p^* \cdot (1-p^*)}{n}} \right]$$ Note the constant 1

Example 1

Before an election 1023 people are asked if they will vote ”yes” or ”no”. 418 say “yes”, 501 say “no” and 104 say “don´t know”. What is the probability of all voters to vote ”yes”?

The case is a binomial distributed random sample. And we choose to calculate the ca. 95% confidence interval:

$$\left[p^* - 2\sqrt{\frac{p^*\cdot(1-p^*)}{n}} \;\;;\;\; p^* + 2\sqrt{\frac{p^*\cdot(1-p^*)}{n}} \right]$$

Here is n = 1023 and the point-estimate $p^* = \frac{418}{1023} \approx 0.409$ =>

$$\left[0.409 - 2\sqrt{\frac{0.409\cdot(1-0.409)}{1023}} \;\;;\;\; 0.409 + 2\sqrt{\frac{0.409\cdot(1-0.409)}{1023}} \right] = [0.378\,; 0.44]$$

Thus there is a 95% probability that between 37.8% and 44% of all the voters will vote “yes”. However, please note that the group ”don´t know” is big, so that the final resolve may change significantly.

Example 2

A brewery will test if people like a new soft drink. They ask an analysis institute for an investigation with a high confidence.

The institute suggests to ask ca. 500 people and then calculate the 99% confidence. The brewery approves.

In practice 494 people are asked if they like the soft drink, ”yes” or ”no”? 77 say ”yes”.

The case is a binomial distributed random sample. And we calculate the ca. 99% confidence interval:

$$\left[p^* - 3\sqrt{\frac{p^*\cdot(1-p^*)}{n}} \;\;;\;\; p^* + 3\sqrt{\frac{p^*\cdot(1-p^*)}{n}} \right]$$

Here is n = 494 and the point-estimate $p^* = \frac{77}{494} \approx 0.156$ =>

$$\left[0.156 - 3\sqrt{\frac{0.156\cdot(1-0.156)}{494}}\ ;\ 0.156 + 3\sqrt{\frac{0.156\cdot(1-0.156)}{494}}\right] = [0.107\,;0.205]$$

Thus there is a 99% probability that between 10.7% and 20.5% of all people may like the soft drink.

A further investigation could be to ask how many would possibly buy the soft drink - and how often.

Predictions in the form of probability calculations and confidence intervals are often correct – but not always. A particularly sensitive point is to take the "right", representative sample.

Notations and technical terms

As previously mentioned there are many technical terms and various notations within probability calculation. It is probably due to different approaches in various professions. Here are the most common ones:

n usually stands for the number in the whole population/pool/quantity.

r usually stands for a chosen number in a population/pool/quantity.

P usually stands for Permutation – but may *also* stand for the Probability function – or just probability.

p usually stands for the probability parameter which is a number between 0 and 1, where p is the probability of success.

K usually stands for combination and has the formula $\frac{n!}{r!\cdot(n-r)!}$

K with the same formula $\frac{n!}{r!\cdot(n-r)!}$ is also used in formulas for the binomial distribution. Then we may write K(n, r) meaning K used in a binomial distribution with the overall number n and the chosen number r. Or we may write $\binom{n}{r}$ *which here is not a vector.*

i.e. $$K(n, r) = \binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$$

x usually stands for a variable.

X usually stands for a stochastic variable (= binomial variable).

$\bar{x} = x_{mid}$ usually stands for the mean value of x

μ usually stands for the mean value of something, which then must be explained. Often, it is the same as $\bar{x}$ or x_{mid}

If something is binomial distributed, it may be written bin(n, p) or just b(n, p) which means: binomial distribution with the number n and the probability parameter p - *and here it is not coordinates of a point.*

σ usually stands for the standard deviation.

* or ^ (star or hat) are often used for sizes in a random sample, for instance μ* and σ* and p*

p* is called the probability parameter of the random sample - or the central estimator of p

Numbers, and brief on set theory

The grouping of numbers was following the historic development of the four types of arithmetic operations:

The *natural numbers* are whole positive numbers - those we count: 1, 2, 3, 4,…

The *whole numbers* are negative, zero, and positive: …-2, -1, 0, 1, 2, 3,…

The *rational numbers* are fractions of whole numbers (except 0 in the denominator). For instance $\frac{-4}{3}$ and $\frac{63}{17}$

The *irrational numbers* cannot be written as fractions of whole numbers but as decimal numbers that never ends. For instance $\pi = 3{,}14\ldots.$

Today these four groups are gathered as ***the real numbers***, ***R.***

From here, there is a sharp border to numbers that are not real, numbers we must imagine, the *imaginary numbers*. The imaginary numbers have been described in chapter: "Imaginary numbers, briefly". Here we repeat:

$\sqrt{-64} = \sqrt{(-1) \cdot 64} = \sqrt{(-1)} \cdot \sqrt{64}$ that is quite ok

$\sqrt{(-1)}$ we name I, that is also allowed, and then we have

$\mathrm{I} \cdot \sqrt{64} = \mathrm{I} \cdot 8$

So $\sqrt{-64} = \mathrm{I} \cdot 8$

which enables us to continue as if nothing has happened; only now, we are in the world of imaginary numbers. Yet, the calculation rule: $\sqrt{a} \cdot \sqrt{a} = \sqrt{a \cdot a}$ does not apply for imaginary numbers.

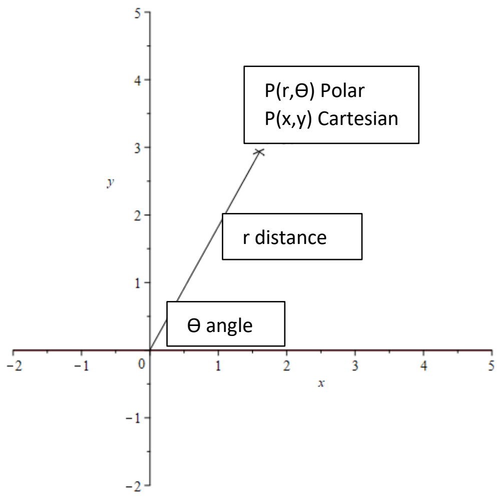
The combination of real numbers and imaginary numbers is called *complex numbers*.

Complex numbers

The combination of real numbers and imaginary numbers is called *complex numbers*.

A brief introduction:

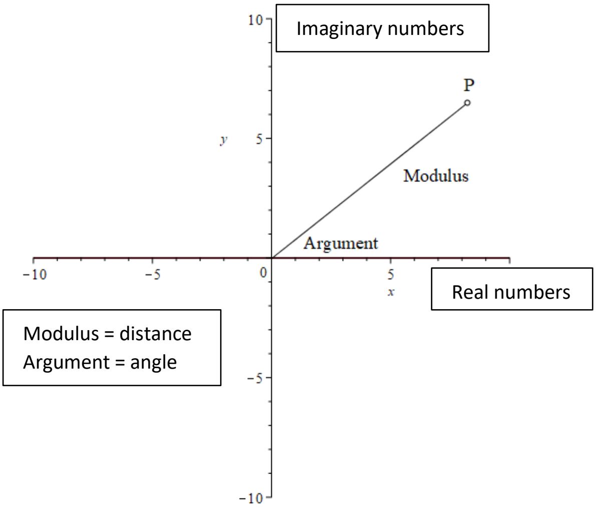
Earlier we saw that coordinates may be shown either in the usual way (Cartesian coordinates) or as polar coordinates, and we repeat a former displayed diagram:



Furthermore, we saw that the coordinates of a point may be shown as a position vector.

Now we will consider the coordinates of a point combined with complex numbers. It looks the same.

We may use the complex numbers as a mathematical tool and display them in a coordinate system with the real numbers on the first axis, and the imaginary numbers on the second axis:



The distance from Origo is now called *modulus* and the angle from the +first axis is called the *argument*. Thus P´s position is

modulus and argument = $|OP|$ *and arg(OP)*

The magnitude we want to describe here (shown as the line segment OP) thus consists of a real part and an imaginary part, and just like for polar coordinates we describe length and angle, only now called modulus and argument, to show that we are considering complex numbers.

Overall we now have several ways to show coordinates. From earlier we have:

- Ordinary Cartesian coordinates P(x,y)
- Position vector coordinates $\mathbf{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} |OP|\cdot\cos\theta \\ |OP|\cdot\sin\theta \end{pmatrix}$

  or $\mathbf{OP} = |OP|\cdot(\cos\Theta\cdot\mathbf{i} + \sin\Theta\cdot\mathbf{j})$

  or short $\mathbf{OP} = x\mathbf{i} + y\mathbf{j}$

  (here it is a little distracting, that the base vector in the x-direction is called **i**, while the imaginary base number on the y-axis is called I (some use i)).

- Polar coordinates P(r,ϴ) distance and angle

And the new complex polar coordinates:

- (mod, arg) or mod∠arg distance and angle

  and an example:
  (mod, arg) = $(5, \frac{\pi}{4})$ or just $5\angle\frac{\pi}{4} = 5\angle 45°$
  The latter is probably the most common.

If the angle is $90° = \frac{\pi}{2}$ we have a purely imaginary number.

It is the most common, and the easiest, to use the vector tool for calculation of complex numbers (also called the rectangular form). Here are some examples using the four basic arithmetic:

Calculation with complex numbers in the rectangular form (as vectors)

Example 1

We have two complex sizes written as

complex number = real part + imaginary part

here: a = 3 + 4I b = -2 + 5I

Sum

a + b = (3 + 4I) + (-2 + 5I) ⇔

a + b = 1 + 9·I separately real and separately imaginary

Difference

$a - b = (3 + 4I) - (-2 + 5I) \quad \Leftrightarrow$

$a - b = 5 - I$

Product

$a \cdot b = (3 + 4I) \cdot (-2 + 5I) \quad \Leftrightarrow$

$a \cdot b = -6 + 15I - 8I + 20I^2 \quad \Leftrightarrow$

$a \cdot b = 20I^2 + 7I - 6 \quad =>$ and since $I = \sqrt{(-1)}$

$a \cdot b = -20 + 7I - 6 \quad \Leftrightarrow$

$a \cdot b = -26 + 7 \cdot I$

Division

$$\frac{a}{b} = \frac{3 + 4I}{-2 + 5I} \quad \Leftrightarrow$$

$$\frac{a}{b} = \frac{(3 + 4I)(-2-5I)}{(-2 + 5I)(-2-5I)} \quad \Leftrightarrow$$

$$\frac{a}{b} = \frac{-6-15I-8I-20I^2}{4+10I-10I-25I^2} \quad \Leftrightarrow$$

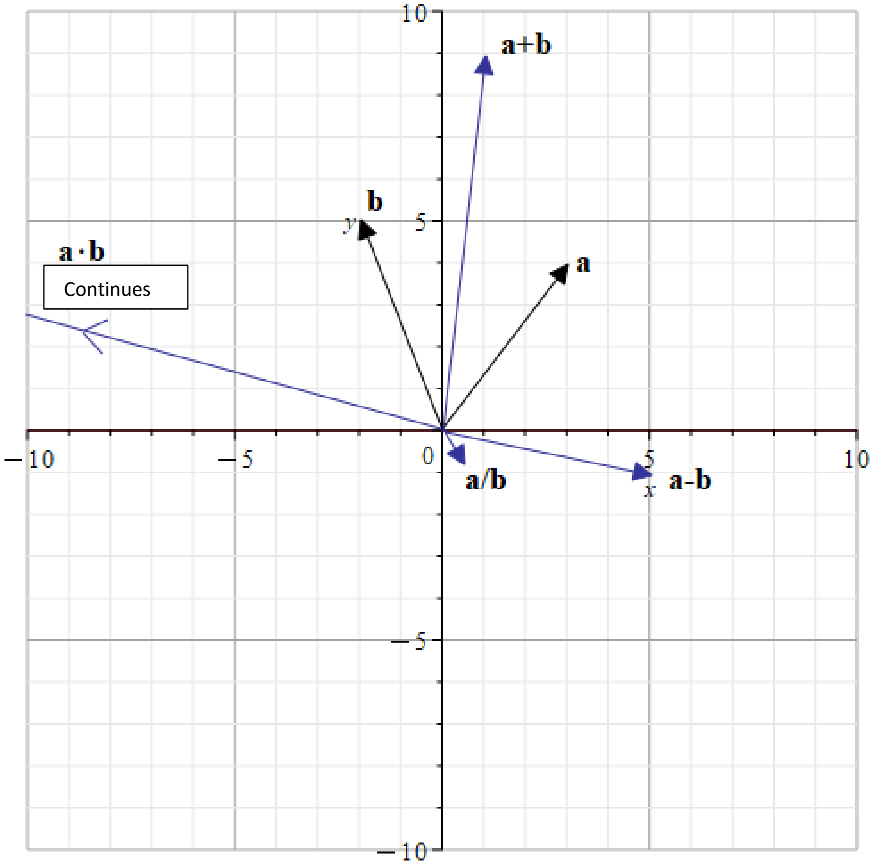
$$\frac{a}{b} = \frac{-6-23I+20}{4+25} \quad \Leftrightarrow$$

$$\frac{a}{b} = \frac{14-23I}{29} \quad \Leftrightarrow$$

$$\frac{a}{b} \approx 0{,}483 - 0{,}793 \cdot I$$

In other words, completely ordinary calculation rules.

and shown in a diagram:



2.

From the rectangular form (vector form) we can find modulus (length) and argument (angle with +x-axis) using well known ordinary calculation:

a = 3 + 4I modulus: $(3^2 + 4^2)^{½} = 5$ (Pythagoras)

and argument: $\tan^{-1}\left(\frac{4}{3}\right) \approx 53{,}1°$

i.e. (mod a, arg a) = (5, 53.1°)

and

$b = -2 + 5I$ has modulus: $((-2)^2 + 5^2)^{½} \approx 5.39$

and argument: $90° + \tan^{-1}(\frac{2}{5}) \approx 111.8°$

i.e. (mod b, arg b) = (5.39, 111.8°)

3.

We can also find modulus and argument

$a \cdot b = -26 + 7 \cdot I$ => modulus $= ((-26)^2 + 7^2)^{½} \approx 26.9$

argument $= 90° + \tan^{-1}(\frac{26}{7}) \approx 164.9°$

i.e. (mod, arg) = (26.9, 164.9°)

$\frac{a}{b} \approx 0{,}48 - 0{,}79 \cdot I$ => modulus $= (0.48^2 + (-0.79)^2)^{½} \approx 0.928$

argument $= \tan^{-1}(\frac{-0.79}{0.48}) \approx -58.7°$

i.e. (mod, arg) = (0.928, -58.7°)

We see that the calculated modulus and argument complies with the diagram.

Calculation with complex numbers in the polar form

Sum and difference of two complex numbers is as for the rectangular form, while product and division instead may be done in the *polar form* which is quicker:

Now we present some calculation rules that are easy to use but heavy to prove. The proof is shown in the chapter ”Rarely used proofs and calculations”:

When we multiply two complex numbers in the polar form, we have:

(mod a , arg Ѳ) · (mod b , arg φ) = (|a| ∠ Ѳ) · (|b| ∠ φ) =

|a·b| ∠ (Ѳ+φ)

so we multiply the modulus (the magnitudes) and add the arguments (the angles).

When we divide two complex numbers in the polar form, we have:

$$\frac{(\text{mod } a,\ \text{arg } \theta)}{(\text{mod } b,\ \text{arg } \varphi)} = \frac{|a|\angle\theta}{|b|\angle\varphi} = \left|\frac{a}{b}\right| \angle(\theta-\varphi)$$

so we divide the modulus (the magnitudes) and subtract the arguments (the angles).

Example 1

Two complex numbers are

(mod, arg) = $(5, \frac{\pi}{4})$ and (mod, arg) = $(3, \frac{\pi}{2})$

We want the product (the two complex numbers multiplied) written polar:

$$\text{modulus} = 5\cdot 3 = 15 \qquad \text{argument} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

Answer: (mod, arg) = $(15, \frac{3\pi}{4})$ or shorter $15\angle\frac{3\pi}{4}$

Regardless of the way we write it (the notation) we multiply the modulus and add the arguments.

2.

And then we want $(5, \frac{\pi}{4})$ divided by $(3, \frac{\pi}{2})$ written polar:

modulus = $\frac{5}{3}$ argument = $\frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

Answer: (mod, arg) = $(\frac{5}{3}, -\frac{\pi}{4})$ or short $\frac{5}{3} \angle -\frac{\pi}{4}$

Regardless of the way we write it (the notation) we divide the modulus and subtract the arguments.

This way of calculating is similar to what we do with exponential functions. Therefore, we may use the exponential function too, when we multiply and divide:

Calculation with complex numbers in the exponential form

Sum and difference of two complex numbers is as for the rectangular form, while product and division instead may be done in the *exponential form* which is used within some industries:

For the polar form, we have just seen, that a product is found by multiplying the modulus (the magnitudes) and adding the arguments (the angles), - while in division we divide the modulus and subtract the arguments. That way of calculating fits well with the exponential function:

$f(x) = b \cdot a^{kx}$ which here becomes $z = |z| \cdot e^{I\Theta}$

(Many tables use z as the complex number, especially in the exponential form).

z is now our complex number, which consists of modulus |z| (the magnitude) and the argument Θ (the angle), which is inserted in the exponential number $e^{I\Theta}$ (with I for Imaginary).

The examples show how we can use this:

Example 1

Two complex numbers (the same as before) are give as

(mod a, arg a) = $(5, \frac{\pi}{4})$ and (mod b, arg b) = $(3, \frac{\pi}{2})$

We want the product (the two complex numbers multiplied) computed using the exponential formula:

$z = |z| \cdot e^{I\Theta}$

Here $a = 5 \cdot e^{I \cdot \frac{\pi}{4}}$ and $b = 3 \cdot e^{I \cdot \frac{\pi}{2}}$ =>

$$a \cdot b = (5 \cdot e^{I \cdot \frac{\pi}{4}}) \cdot (3 \cdot e^{I \cdot \frac{\pi}{2}}) = 15 \cdot e^{I \cdot \frac{\pi}{4} + I \cdot \frac{\pi}{2}} = 15 \cdot e^{I \cdot \frac{3\pi}{4}}$$

We see that the modulus is 15 and the argument is $\frac{3\pi}{4}$

in short: (mod, arg) = $(15, \frac{3\pi}{4})$ or very short $15\angle \frac{3\pi}{4}$

Same answer as for the polar form. Of course.

2.

Two complex numbers are given as

(mod a, arg a) = $(5, \frac{\pi}{4})$ and (mod b, arg b) = $(3, \frac{\pi}{2})$

We want to divide according to the exponential formula:

$z = |z| \cdot e^{I\Theta}$

Here $\quad a = 5 \cdot e^{I \cdot \frac{\pi}{4}} \quad$ and $\quad b = 3 \cdot e^{I \cdot \frac{\pi}{2}} \quad$ =>

$$\frac{a}{b} = \frac{(5 \cdot e^{I \cdot \frac{\pi}{4}})}{(3 \cdot e^{I \cdot \frac{\pi}{2}})} = \frac{5}{3} \cdot e^{I \cdot \frac{\pi}{4} - I \cdot \frac{\pi}{2}} = \frac{5}{3} \cdot e^{I \cdot (-\frac{\pi}{4})}$$

We see that the modulus is $\frac{5}{3}$ and the argument is $-\frac{\pi}{4}$

in short: $\quad$ (mod, arg) = $(\frac{5}{3}, -\frac{\pi}{4})$ or very short $\frac{5}{3} \angle -\frac{\pi}{4}$

Same answer as for the polar form. Of course.

Summary

Complex numbers are not common but they may be used as a mathematical tool within electronics, advanced description of flowing liquids, etc.

The rectangular form can be used in all four basic arithmetic operations.

The polar form is quick when multiplying or dividing.

The exponential form is used in some industries for multiplication and division.

Example

We will finish by continuing an earlier example and have a survey of the three methods:

<u>Two complex numbers in the rectangular form:</u>

a = 3 + 4I and b = -2 + 5I

Sum a + b = (3 + 4I) + (-2 + 5I) = 1 + 9·I

Difference a - b = (3 + 4I) - (-2 + 5I) = 5 - I

Product a · b = (3 + 4I) · (-2 + 5I) = -26 + 7·I

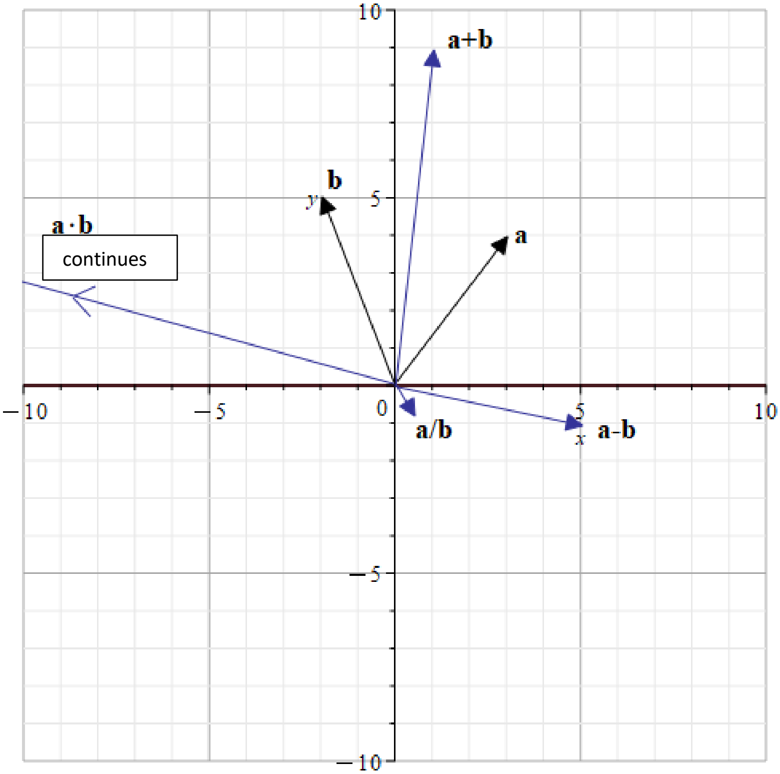
Division $\frac{a}{\mathrm{b}} = \frac{3 + 4\mathrm{I}}{-2 + 5\mathrm{I}} \approx 0{,}483 - 0{,}793 \cdot \mathrm{I}$

$|a| = (3^2 + 4^2)^{½} = 5$ $\tan^{-1}(\frac{4}{3}) \approx 53{,}1°$

$|b| = ((-2)^2 + 5^2)^{½} \approx 5.39$ $90° + \tan^{-1}(\frac{2}{5}) \approx 111{,}8°$

Both arguments (angles) are relative to the positive direction of the first axis.

The complex numbers a and b are shown as vectors in the diagram, - which we show again:



Two complex numbers in the polar form:

The calculation of modulus and argument for the rectangular form are now used in the polar form:

$a \approx 5 \angle 53.1°$ and $b \approx 5.39 \angle 111.8°$

We cannot add or subtract in the polar form, but we can multiply and divide:

$a \cdot b \approx 5 \cdot 5.39 \angle (53.1° + 111.8°) \approx 26.9 \angle 164.9°$

$\frac{a}{b} \approx \frac{5}{5.39} \angle (53.1° - 111.8°) \approx 0.929 \angle -58.7°$

-58.7° may also be written +301.3°

It is seen that both comply with the earlier calculations as well as with the figure.

Two complex numbers in the exponential form

The calculations of modulus and argument for the rectangular form are now used in the exponential form:

a $\approx$ 5 $\angle$53.1° and b $\approx$ 5.39 $\angle$111.8°

yet, in the exponential form it is common to use radians:

a $\approx$ 5 $\angle$0.927 and b $\approx$ 5.39 $\angle$1.95

We cannot add or subtract in the exponential form, but we can multiply and divide:

$$a \cdot b \approx (5 \cdot e^{I \cdot 0.927}) \cdot (5.39 \cdot e^{I \cdot 1.95}) \approx 26.95 \cdot e^{I \cdot 0.927 + I \cdot 1.95}$$

$$\Leftrightarrow \quad a \cdot b \approx 26.95 \cdot e^{I \cdot 2.88}$$

We see that modulus is 26.95 and argument is 2.88

Answer: (mod, arg) = (26.95, 2.88) or short 26.95 $\angle$2.88

$$\frac{a}{b} \approx \frac{5 \cdot e^{I \cdot 0.927}}{5.39 \cdot e^{I \cdot 1.95}} \approx 0.929 \cdot e^{I \cdot 0.927 - I \cdot 1.95} \approx 0.928 \cdot e^{I \cdot (-1.02)}$$

We see that modulus is 0.928 and argument is -1.02

Answer: (mod, arg) = (0.929, -1.02) or short 26.95 $\angle$- 1.02

-1.02 rad. can also be written +5.26 rad.

Brief on set theory

The set theory is only to be mentioned briefly. In part because it is an earlier acquired part of mathematics, and in part because most tables present the signs of the set theory with explanatory sketches.

Here, we emphasize:

The empty quantity (the empty set, nothing) is written Ø

The solution set is written {-,-,-,…} or by using a letter which is different in various countries.

ϵ means ”belongs to” or “is an element of the set”.

$\wedge$ means and.

$\vee$ means or.

Examples

If, in a problem, we are informed that the solution must belong to the real numbers, it may be written as $x \epsilon R$

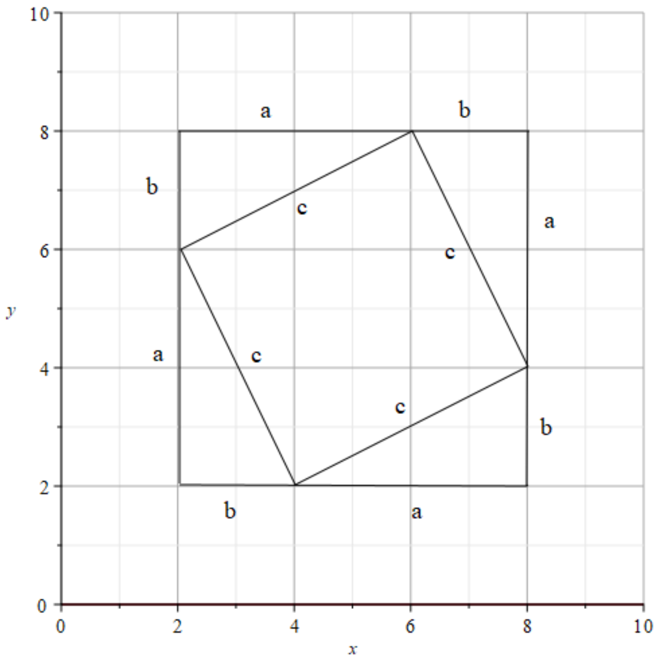
If, in a solve, we reach: *no solution* we also may write *the solution is the empty quantity, Ø*

If, in a problem, we solve to get a domain for the function f in the interval]-50; 0] we may write:
The domain of f is]-50; 0] or brief *Dm(f) =]-50; 0]*

If we want to display that x belongs to the interval]-50; 0] , we may write $x \epsilon\ ]-50; 0]$

Rarely used proofs and calculations

Proof of Pythagoras theorem



Two squares are shown in the diagram.

The big has the area $(a + b)^2$

The small has the area c^2

The area of the big square equals the area of the small square plus the areas of the four triangles:

$$(a + b)^2 = c^2 + 4 \cdot \frac{1}{2} \cdot a \cdot b \qquad \Leftrightarrow$$

$$a^2 + b^2 + 2ab = c^2 + 2ab \qquad \Leftrightarrow$$

$$a^2 + b^2 = c^2$$

Hence we have proved the probably most common mathematical theorem.

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Proof of factorization of a second degree polynomium

$ax^2 + bx + c = a(x - root_1)(x - root_2)$

We prove by calculating from the right side (the resolve) back to the left side (the starting point), knowing that:

$$root_1 = \frac{-b+\sqrt{d}}{2a} \quad and \quad root_2 = \frac{-b-\sqrt{d}}{2a}$$

which is inserted to the right:

$a\left(x - \frac{-b+\sqrt{d}}{2a}\right)\left(x - \frac{-b-\sqrt{d}}{2a}\right)$	=	signs arranged
$a\left(x + \frac{b-\sqrt{d}}{2a}\right)\left(x + \frac{b+\sqrt{d}}{2a}\right)$	=	multiplication
$a\left(x^2 + x\frac{b+\sqrt{d}}{2a} + x\frac{b-\sqrt{d}}{2a} + \frac{b^2-d}{4a^2}\right)$	=	common denominator
$a\left(\frac{4a^2x^2+2axb+2ax\sqrt{d}+2axb-2ax\sqrt{d}+b^2-d}{4a^2}\right)$	=	
$a\left(\frac{4a^2x^2+4axb+b^2-(b^2-4ac)}{4a^2}\right)$	=	shortening
$a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$	=	multiplication
$ax^2 + bx + c$		which is the left side

Hence proven that:

$$ax^2 + bx + c = a(x - root_1)(x - root_2)$$

There is also proof of factorization of higher polynomials, but we stop here.

Division of polynomials

We can divide a polynomial by another polynomial using a technique similar to ordinary division - but more complicated. The technique is seen in this example:

Example 1

$\frac{x^4-2x^3-3x^2+12x-18}{x^2-6}$ calculates this way:

$$x^2-6 \;\rfloor\; x^4-2x^3-3x^2+12x-18 \;\lfloor\;\underline{\qquad\qquad}$$

First we focus on x^4 divided by x^2. This gives x^2 which is written to the right. Then we multiply x^2 by $(x^2 - 6)$. That gives $x^4 - 6x^2$ which is written in the next line:

$$\begin{array}{l|l|l} x^2-6 & x^4-2x^3-3x^2+12x-18 & x^2 \\ & x^4 \qquad\quad -6x^2 & \end{array}$$

Then we say upper minus lower and divide $-2x^3$ by x^2. This gives $-2x$ which is written in the resolve to the right. Then we multiply $-2x$ by $(x^2 - 6)$ and write the answer in the next line – followed by upper minus lower:

$$\begin{array}{l|l|l} x^2-6 & x^4-2x^3-3x^2+12x-18 & x^2-2x \\ & \underline{x^4 \qquad\quad -6x^2} & \\ & \quad -2x^3+3x^2+12x-18 & \\ & \quad \underline{-2x^3 \qquad\quad +12x} & \end{array}$$

$$3x^2 \qquad\qquad - 18$$

We divide $3x^2$ by x^2. This gives 3 which is written in the resolve to the right. Then we multiply 3 by $(x^2 - 6)$ and write the answer in the next line:

$$
\begin{array}{l|lllll|l}
x^2 - 6 & x^4 & - 2x^3 & - 3x^2 & + 12x & - 18 & x^2 - 2x + 3 \\
 & x^4 & & - 6x^2 & & & \\
\hline
 & & -2x^3 & + 3x^2 & + 12x & - 18 & \\
 & & -2x^3 & & + 12x & & \\
\hline
 & & & 3x^2 & & - 18 & \\
 & & & 3x^2 & & - 18 & \\
\hline
 & & & 0 & & 0 & \\
\hline
\end{array}
$$

Upper minus lower gives 0 and we are done. It added up.

The resolve is $\qquad x^2 - 2x + 3$

Example 2

Same technique if it doesn't add up. Then we get a remainder:

$\frac{x^4-2x^3-3x^2+12x-17}{x^2-6}$ which calculates similarly:

$$
\begin{array}{l|l|l}
\underline{x^2 - 6} & x^4 - 2x^3 - 3x^2 + 12x - 17 & \underline{x^2 - 2x + 3} \\
 & \underline{x^4 \qquad\quad - 6x^2} & \\
 & \qquad -2x^3 + 3x^2 + 12x - 17 & \\
 & \qquad \underline{-2x^3 \qquad\quad + 12x \quad} & \\
 & \qquad\qquad\qquad 3x^2 \qquad\quad -17 & \\
 & \qquad\qquad\qquad \underline{3x^2 \qquad\quad -18} & \\
 & \qquad\qquad\qquad\qquad\qquad\quad 1 &
\end{array}
$$

1 is the remainder which also must be divided by $(x^2 – 2x)$.

Combined resolve $\quad x^2 - 2x + 3 + \frac{1}{x^2-6}$

We rarely need division of polynomials, and if we do, we will often use CAS. Yet, not all CAS is capable, especially if it does not add up, then we must do it manually.

Showing the formulas for Permutation and Combination

For Permutation we use example 3 from the chapter on Probability:

3.

A foreman and deputy foreman and alternate must be elected in a board with 7 members. The one first elected becomes foreman, the next becomes deputy foreman, and the third becomes alternate. In how many ways can the 3 people be elected?

Since the elected is not put back into the pool, the case is *any order, without repetition*.

There are 7 possibilities of selecting the first, 6 possibilities of selecting the next, and 5 possibilities of selecting the third and last.

The case is *”Both, and”* i.e. multiplication: $7 \cdot 6 \cdot 5 = 210$ possibilities

Since the selection is not random all possibilities will be different, and there are 210 possibilities.

Now we want a formula including the 3 selected as well as the 7 members of the pool/population, since this is our introductory information.

We choose to multiply the number of possibilities $7 \cdot 6 \cdot 5$ with $4 \cdot 3 \cdot 2 \cdot 1$ which is allowed if we also divide by $4 \cdot 3 \cdot 2 \cdot 1$. That is:

$\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{7!}{4!} = \frac{7!}{(7-3)!} = \frac{n!}{(n-r)!} = P$ hereby the formula is shown

Thus we have the number in the pool/population (here the number of members 7) which we call n. And we have the number selected (here 3) which we call r.

For Combination we use example 4 from the chapter on Probability:

4.

A foreman and deputy foreman and alternate must be elected in a board with 7 members. The election will show who of the 3 people are having the 3 positions – regardless of which position. The decision amongst the 3 is postponed until later. How many possibilities are there for selection of the 3 people?

Here the order does not matter, and the one selected is not put back into the pool. Thus the case is *no order, without repetition.*

There are 7 possibilities of selecting the first, 6 possibilities of selecting the next, and 5 possibilities of selecting the third and last.

The case is *"Both, and"* i.e. multiplication: $7 \cdot 6 \cdot 5 = 210$ possibilities

Since the selection is random some possibilities will be alike. How many is that? Let us name the people:

Ann, Ben, Clara, Dan, Ellie, Fred

Then

Ann, Ben, Clara = Ben, Clara, Ann = Clara, Ben, Ann

i.e. three equal selections.

Ben, Clara, Dan   will also give three equals.

Clara, Dan, Ellie   will also give three equals.

Dan, Ellie, Fred   will also give three equals.

Ellie, Fred, Ann   will also give three equals.

Fred, Ann, Ben   will also give three equals.

So, 18 ways are actually only 6 possibilities. Let us call them 6 "packages".

Or put in another way: The 3 people may swap in $6 = 3 \cdot 2 \cdot 1$ ways which renders:

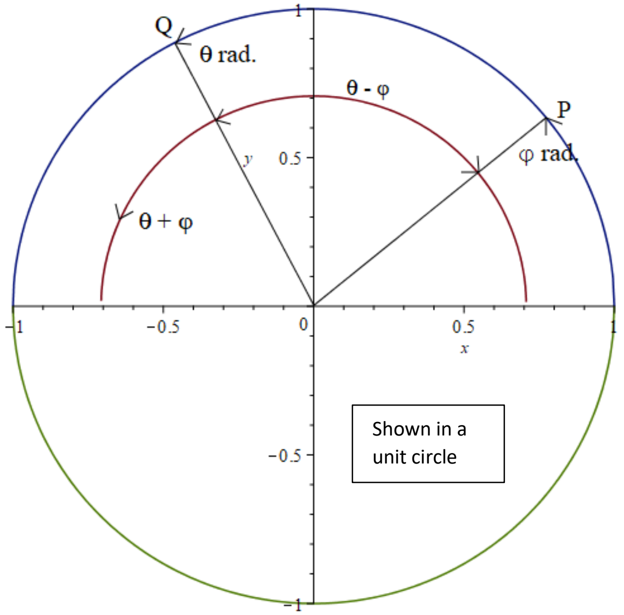
$\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35 \ \ real\ possibilities$

We now prolong this fraction by $4 \cdot 3 \cdot 2 \cdot 1$ in numerator and denominator and built in the size of the population (here 7) which we call n - and we also built in the number of “packages” (here 6 = 3!) which we call r

$\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7!}{3! \cdot 4!} = \frac{7!}{3! \cdot (7-3)!} = \frac{n!}{\mathrm{r}! \cdot (\mathrm{n}-\mathrm{r})!} = K$ hereby the formula is shown.

Proof for product and division of complex numbers in the polar and the exponential form

First, we need to derive some formulas on conversion of expressions including sine and cosine. There are many of them. Here we will use *the four addition formulas*. We call the angles ϴ and φ. The proof is valid for angles measured in both degrees and radians. Here we use radians:



Point P has the angle φ with the positive part of the first axis.

Point Q has the angle ϴ with the positive part of the second axis.

Angle ϴ-φ is between the two angle legs.

Angle ϴ+φ is from the +x direction to the arrow shown.

The four addition formulas are:

1. cos(ϴ+φ) = cos ϴ·cos φ - sin ϴ·sin φ
2. cos(ϴ-φ) = cos ϴ·cos φ + sin ϴ·sin φ
3. sin(ϴ+φ) = sin ϴ·cos φ + sin φ·cos ϴ
4. sin(ϴ-φ) = sin ϴ·cos φ - sin φ·cos ϴ

No.2 is proved by one of the formulas for an angle between two vectors:

$$\cos v = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|} \qquad \Rightarrow \qquad \text{here}$$

$$\cos(\Theta-\varphi) = \frac{\mathbf{OQ}\cdot\mathbf{OP}}{|\mathbf{OQ}|\ |\mathbf{OP}|} = \frac{\binom{\cos\Theta}{\sin\Theta}\cdot\binom{\cos\varphi}{\sin\varphi}}{1\cdot1} = \cos\Theta\cdot\cos\varphi + \sin\Theta\cdot\sin\varphi$$

No.1 is proved by rearranging no.2

$$\cos(\Theta+\varphi) = \cos(\Theta-(-\varphi)) = \frac{\binom{\cos(\Theta)}{\sin(\Theta)}\cdot\binom{\cos(-\varphi)}{\sin(-\varphi)}}{1\cdot1} = \cos\Theta\cdot\cos(-\varphi) + \sin\Theta\cdot\sin(-\varphi)$$

and since $\cos(-\varphi) = \cos\varphi$ and $\sin(-\varphi) = -\sin\varphi$ (see the unit circle), we have

$$\cos(\Theta+\varphi) = \cos\Theta\cdot\cos\varphi - \sin\Theta\cdot\sin\varphi$$

No.4 is proved by the other formula for an angle between two vectors:

$$\sin v = \frac{\det(\mathbf{a},\mathbf{b})}{|\mathbf{a}|\cdot|\mathbf{b}|} \qquad \Rightarrow \qquad \text{here}$$

$$\sin(\Theta-\varphi) = \frac{\det(\mathbf{OP},\mathbf{OQ})}{|\mathbf{OP}|\cdot|\mathbf{OQ}|} = \frac{\begin{pmatrix}\cos\varphi & \cos\Theta\\ \sin\varphi & \sin\Theta\end{pmatrix}}{1\cdot1} = \cos\varphi\cdot\sin\Theta - \sin\varphi\cdot\cos\Theta$$

No.3 is proved by rearranging no.4

$$\sin(\Theta+\varphi) = \sin(\Theta-(-\varphi)) = \frac{\det(\mathbf{OP},\mathbf{OQ})}{|\mathbf{OP}|\cdot|\mathbf{OQ}|} = \frac{\begin{pmatrix}\cos(-\varphi) & \cos\Theta\\ \sin(-\varphi) & \sin\Theta\end{pmatrix}}{1\cdot1} =$$

$$\cos(-\varphi)\cdot\sin\Theta - \sin(-\varphi)\cdot\cos\Theta$$

and since $\cos(-\varphi) = \cos\varphi$ and $\sin(-\varphi) = -\sin\varphi$ (see the unit circle), we have

$$\sin(\Theta+\varphi) = \cos\varphi\cdot\sin\Theta + \sin\varphi\cdot\cos\Theta$$

The next point is about notation.

Our complex number z may be written as a vector

$z = z_1 + I \cdot z_2$ or $z = |z| \cdot (\cos \Theta + I \cdot \sin \Theta)$

(Both equations say that our complex number has a horizontal part + a vertical, which is: a real part + an Imaginary part)

Below we will get to the form $z = |z| \cdot (\cos \Theta + I \cdot \sin \Theta)$

which will be written in the polar form.

Now we are ready for the next step in the proof of product and division of complex numbers in the polar form:

Product: $(\text{mod.a}, \text{arg.}\Theta) \cdot (\text{mod.b}, \text{arg.}\varphi) = |a \cdot b| \angle(\Theta + \varphi)$

Proof: $(\text{mod.a}, \text{arg.}\Theta) \cdot (\text{mod.b}, \text{arg.}\varphi) = a(\cos \Theta + I \cdot \sin \Theta) \cdot b(\cos \varphi + I \cdot \sin \varphi =$

$ab\,(\cos \Theta \cdot \cos \varphi + \cos \Theta \cdot i \cdot \sin \varphi + I \cdot \sin \Theta \cdot \cos \varphi + I \cdot \sin \Theta \cdot I \cdot \sin \varphi) =$

$ab\,((\cos \Theta \cdot \cos \varphi - \sin \Theta \cdot \sin \varphi) + I(\cos \Theta \cdot \sin \varphi + \sin \Theta \cdot \cos \varphi))$

And by using the addition formulas no.1 and no.3 =>

$ab\,(\cos(\Theta + \varphi) + I \cdot \sin(\Theta + \varphi))$

In the parenthesis we have the angles added, and the coordinates split in a real and an imaginary part. It may be written this way:

$(\text{mod.a}, \text{arg.}\Theta) \cdot (\text{mod.b}, \text{arg.}\varphi) = |a \cdot b| \angle(\Theta + \varphi)$

We multiply the moduli (the magnitudes) and sum the arguments (the angles). The notation shows that we are calculating polar.

Division: $\frac{(\text{mod.a}, \ \text{arg.}\Theta)}{(\text{mod.b}, \ \text{arg.}\varphi)} = |\frac{a}{b}| \angle(\Theta - \varphi)$

Proof: $\frac{(\text{mod.a}, \ \text{arg.}\Theta)}{(\text{mod.b}, \ \text{arg.}\varphi)} = \frac{a(\cos \Theta + I \cdot \sin \Theta)}{b(\cos \varphi + I \cdot \sin \varphi)}$ prolonged =

$\frac{a(\cos \Theta + I \cdot \sin \Theta) \cdot (\cos \varphi - I \cdot \sin \varphi)}{b(\cos \varphi + I \cdot \sin \varphi) \cdot (\cos \varphi - I \cdot \sin \varphi)}$ multiplied =

$\frac{a(\cos \Theta \cdot \cos \varphi - \cos \Theta \cdot I \cdot \sin \varphi + I \cdot \sin \Theta \cdot \cos \varphi + \sin \Theta \cdot \sin \varphi)}{b((\cos \varphi)^2 - (\cos \varphi \cdot I \cdot \sin \varphi) + I \cdot \sin \varphi \cdot \cos \varphi + (\sin \varphi)^2)}$ arranged =

$$\frac{a(\cos\Theta\cdot\cos\varphi + \sin\Theta\cdot\sin\varphi) + I(\sin\Theta\cdot\cos\varphi - \cos\Theta\cdot\sin\varphi)}{b((\cos\varphi)^2 + (\sin\varphi)^2 + I(\sin\varphi\cdot\cos\varphi - \cos\varphi\cdot\sin\varphi))} \quad =$$

In the first two terms of the numerator we use addition formula no.2 – and in the last two terms we use no.4. In the first two terms of the denominator we use the basic relationship and the last two terms give zero:

$$\frac{a(\cos(\Theta-\varphi) + I\cdot\sin(\Theta-\varphi))}{b(1+0)} = \frac{a}{b}(\cos(\Theta\text{-}\varphi) + I\cdot\sin(\Theta\text{-}\varphi))$$

which is written

$$\frac{(\text{mod.a}\,,\ \text{arg.}\Theta)}{(\text{mod.b}\,,\ \text{arg.}\varphi)} = \left|\frac{a}{b}\right| \angle (\Theta\text{-}\varphi)$$

We divide the moduli (the magnitudes) and subtract the arguments (the angles). The notation shows that we are calculating polar.

Conversion from polar to exponential form was shown in the chapter on complex numbers. Euler put it into a formula by simply equaling one of the vector expressions to the exponential function. Put in another way: he defined the equation:

$$z = |z|\cdot(\cos\Theta + I\cdot\sin\Theta) = z\cdot e^{I\cdot\Theta}$$

Subject index